7) State the Rational Roots Theorem

Let $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{Z}[x]$. If $r = \frac{a}{b}$ is a rational root of $f(x)$, then $a \mid a_0$ and $b \mid a_n$.

Let $f(x) = 4x^4 + 8x^3 + 9x^2 + 5x + 1$. Determine the rational roots of $f(x)$.

Solution: By the theorem, we have that the only possibilities are $r = \pm 1, \pm 1/2, \pm 1/4$. We can plug in each of these values to see that

$$\begin{aligned} f(1) &= 27 \\ f(-1) &= 1 \\ f(1/2) &= 7 \\ f(-1/2) &= 0 \\ f(1/4) &= 189/64 \\ f(-1/4) &= 13/64 \end{aligned}$$

Hence only $-1/2$ is a rational root of $f(x)$.

Factor $f(x) = 4x^4 + 8x^3 + 9x^2 + 5x + 1$ over $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$ and $\mathbb{Z}_5[x]$.

By the above, we know that $2x + 1$ is a factor of $f(x)$ using the Factor Theorem. Hence, using long division:

$$\begin{array}{r|l} & 2x^3 + 3x^2 + 3x + 1 \\ \hline 2x+1 & 4x^4 + 8x^3 + 9x^2 + 5x + 1 \\ & \underline{4x^4 + 2x^3} \\ & \quad 6x^3 + 9x^2 \\ & \quad \underline{6x^3 + 3x^2} \\ & \qquad 6x^2 + 5x \\ & \qquad \underline{6x^2 + 3x} \\ & \qquad\quad 2x + 1 \\ & \qquad\quad \underline{2x + 1} \\ & \qquad\qquad 0 \end{array}$$

we have that $f(x) = (2x+1)(2x^3+3x^2+3x+1)$. From the previous part we know that the only possible rational root that $2x^3+3x^2+3x+1$ can have is once again $-1/2$. We close our eyes and hope that this is a root:

$$2(-1/2)^3 + 3(-1/2)^2 + 3(-1/2) + 1 = -2/8 + 3/4 - 3/2 + 1 = 2/4 - 3/2 + 1 = 0$$

Huzzah! Thus, another long division

$$\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
2x+1 & 2x^3 + 3x^2 + 3x + 1 \\
 & \underline{2x^3 + x^2} \\
 & \qquad 2x^2 + 3x \\
 & \qquad \underline{2x^2 + x} \\
 & \qquad\qquad 2x + 1 \\
 & \qquad\qquad \underline{2x + 1} \\
 & \qquad\qquad\qquad 0
\end{array}$$

implies that $f(x) = (2x+1)^2(x^2+x+1)$. The roots of this last equation are given by the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

These roots are imaginary and thus the factorization over $\mathbb{Q}$ and $\mathbb{R}$ is given by

$$f(x) = (2x+1)^2(x^2+x+1)$$

and over $\mathbb{C}$ by

$$f(x) = (2x+1)^2(x - (1+\sqrt{3}i)/2)(x - (1+\sqrt{3}i)/2).$$

Finally, over $\mathbb{Z}_5$, notice that if the last term factors, it must have a rational root (this was Q7 of the last homework!) By the factor theorem, it must have a root if it factors.

Testing the five values gives:

$$\begin{aligned}
0^2+0+1&=1\\
1^2+1+1&=3\\
2^2+2+1&=2\\
3^2+3+1&=3\\
4^2+4+1&=1
\end{aligned}$$

and thus it has no root. Hence over $\mathbb{Z}_5$, the factorization is $f(x)=(2x+1)^2(x^2+x+1)$.