4) If $a, b, c \in \mathbb{Z}$ with $c \mid ab$ and $\gcd(a, c) = 1$ then $c \mid b$.

Solution: Since $\gcd(a, c) = 1$, we may apply Bezout's Lemma (EEA) to see that there exist integers x, y such that

$$ax + cy = 1$$

Multiplying through by b gives

$$abx + cby = b$$

Since $c \mid ab$ and $c \mid cby$, by DIC we see that $c \mid b$ as required.

Prove: For integers a, b, c such that $\gcd(a, b) = 1$ if $a \mid c$ and $b \mid c$ then $ab \mid c$.

Solution: By definition, there exist integers m and n such that $am = c$ and $bn = c$. Then, by Bezout's Lemma,

$$ax + by = 1$$

Multiplying by mn yields

$$amnx + bmny = mn$$

Substituting yields

$$cnx + cmy = mn$$

Hence by DIC, $c \mid mn$ so $ck = mn$ for some integer k. Now, $am = c$ implies that $amk = ck = mn$ and thus, $ak = n$ (or $m = 0$ in which case $c = 0$ and the claim holds). Multiplying by b yields $abk = bn = c$ and hence $ab \mid c$ as required.