The Riemann-Roch Theorem

by

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Declaration

I hereby declare that I am the sole author of this project. This is a true copy of the project, including any required final revisions, as accepted by my examiners.

I understand that my project may be made electronically available to the public.

Abstract

In this paper, I present varied topics in algebraic geometry with a motivation towards the Riemann-Roch theorem. I start by introducing basic notions in algebraic geometry. Then I proceed to the topic of divisors, specifically Weil divisors, Cartier divisors and examples of both. Linear systems which are also associated with divisors are introduced in the next chapter. These systems are the primary motivation for the Riemann-Roch theorem. Next, I introduce sheaves, a mathematical object that encompasses a lot of the useful features of the ring of regular functions and generalizes it. Cohomology plays a crucial role in the final steps before the Riemann-Roch theorem which encompasses all the previously developed tools. I then finish by describing some of the applications of the Riemann-Roch theorem to other problems in algebraic geometry.

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Dedication

To my family and friends.

Contents

Chapter 1

Introduction

Algebraic geometry is the study of geometric principles using algebraic ideas. Through this approach, many new results can be derived and many innovative ideas have been developed by using correspondences between algebra and geometry. Ideas such as divisors, sheaves, and sheaf cohomology can all be found to have underlying geometric roots despite being very algebraic concepts.

In this paper, I wish to describe one very important connection between algebra and geometry called the Riemann-Roch theorem. This useful theorem of arithmetic geometry gives us an idea of how topological, algebraic and geometric ideas can all come together to form a beautiful succinct formula. It encompasses many of the ideas from divisors and sheaves and results from very straightforward applications of these intricate concepts.

I will begin by introducing some of the basic terminology from algebraic geometry that will serve as a foundation for subsequent chapters. I will then discuss Weil divisors followed by Cartier divisors and their connections. Linear systems are the next natural step towards the Riemann-Roch theorem and are indeed one of the more motivating structures towards the inception of this theorem. Next, I will introduce the powerful language of sheaves and discuss how divisors bring about special kinds of sheaves. I will also introduce the canonical class associated with varieties. I then delve into Čech cohomology which I will use strictly as a device geared towards the proof of the Riemann-Roch theorem. As such, I will not fully develop the theory but rather explain the notation and terminology and then reference many of the major results that I will use to prove the Riemann-Roch theorem. Equipped with our armamentarium of mathematical tools, I will present the Riemann-Roch theorem in its entirety followed by some of its many applications to algebraic geometry.

Chapter 2

Introductory Algebraic Geometry

In this chapter, I will go through some of the preliminary notions that are fundamental in the study of algebraic geometry. In general, our primary focus will be with smooth projective varieties but whenever possible, I will present theorems in their most general form. I start with reviewing the notions of a variety and then proceed into projective varieties. I will define what it means to take the dimension of a variety and I will talk about the local ring of a variety X with respect to a subvariety Y. This turns out to be a discrete valuation ring whenever the codimension of Y is 1 and in the next chapter, I will define a valuation on it to help describe properties with divisors.

Throughout this paper, let k be an algebraically closed field.

2.1 Affine Geometry

Definition 2.1. Let $\mathbb{A}^n := \{(a_1, .., a_n) \mid a_i \in k\}$. A subset $X \subseteq \mathbb{A}^n$ is an *algebraic set* if it is the zero set of a set of polynomials S defined over $k[x_1, .., x_n]$. Denote the zero set by $V(S)$.

Definition 2.2. Let $X \subseteq \mathbb{A}^n$ be a non-empty algebraic set. Suppose that X is such that if $X = X_1 \cup X_2$ for algebraic sets X_1 and X_2 then one of X_1 or X_2 equals X. Then we call X an *irreducible algebraic set.*

Definition 2.3. An *affine algebraic variety* is an irreducible algebraic set. We will systematically identify algebraic varieties over k with their k-rational points.

Definition 2.4. The *ideal of an algebraic set* is $I(X) := \{f \in k[x_1, .., x_n] \mid f(p) = 0 \text{ for all } p \in X\}$. Note that if $X \subseteq Y$ for algebraic sets X, Y, then $I(Y) \subseteq I(X)$.

Definition 2.5. The *coordinate ring* of an affine variety X, denoted by $\Gamma(X)$, is

$$\Gamma(X) := k[x_1, .., x_n]/I(X)$$

Proposition 2.6. *An affine algebraic set X is a variety if and only if $I(X)$ is a prime ideal in $\Gamma(X)$.*

Proof. $\Rightarrow$ Suppose X is a variety and let $f, g \in k[x_1, .., x_n]$ with $fg \in I(X)$. Then $< fg > \subseteq I(X)$. Moreover, $X = V(I(X)) \subseteq V(fg) = V(f) \cup V(g)$. Hence $X = (X \cap V(f)) \cup (X \cap V(g))$. Without loss of generality, the irreducibility of X implies that $X = X \cap V(f) \subseteq V(f)$ and thus $f \in I(X)$ proving that $I(X)$ is prime.

$\Leftarrow$ Suppose that $I(X)$ is prime and suppose $X = X_1 \cup X_2$. This reveals that $I(X) = I(X_1) \cap I(X_2)$. If $I(X) = I(X_1)$ then $X_1 = X$. Otherwise, there exists an $f \in I(X_1) \backslash I(X)$. However, for any $g \in I(X_2)$, it is the case that $fg \in I(X_1) \cap I(X_2) = I(X)$. The primality of $I(X)$ yields that $g \in I(X)$ (since $f \notin I(X)$). Thus $I(X_2) \subseteq I(X) \subseteq I(X_2)$ and hence $I(X) = I(X_2)$ in other words $X = X_2$. Hence X is irreducible as required. ■

2.2 Projective Geometry

Definition 2.7. Define the *projective n-space* $\mathbb{P}^n$ to be

$$\mathbb{P}^n = \mathbb{A}^{n+1} \backslash \{0\} / \sim$$

where $\sim$ is an equivalence relation defined by

$$(x_0, .., x_n) \sim (y_0, .., y_n)$$
$$\Leftrightarrow (x_0, .., x_n) = \lambda(y_0, .., y_n)$$

for some $\lambda \in k^*$.

Remark. In a similar manner to the affine case, we can define a *projective algebraic set*, the *ideal of a projective algebraic set*, and an *irreducible projective algebraic set*.

Definition 2.8. Let $X \subseteq \mathbb{P}^n$. Then X is a *projective variety* if $X = \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in S\}$ where S is a set of homogeneous polynomials of degree d defined over $k[x_0, .., x_n]$. Note that although it does not make sense to evaluate functions at projective points, testing to see whether a function vanishes at a projective point is a well defined notion. This follows by noting for $x := [x_0 : .. : x_n] \in \mathbb{P}^n$ and $s \in k$, one sees that $f(sx) = s^d f(x)$

and so if x is a zero point, then $f(sx) = f(x) = 0$. This is necessary since points in $\mathbb{P}^n$ are only defined up to multiplication by a nonzero scalar. Notice that functions in the fraction field, say $\frac{f}{g}$ are well defined so long as the degree of the numerator equals the degree of the denominator. This occurs because

$$\frac{f(sx)}{g(sx)} = \frac{s^{\deg(f)}f(x)}{s^{\deg(g)}g(x)} = s^{\deg(f)-\deg(g)}\frac{f(x)}{g(x)}$$

Thus, the evaluation map is well defined if and only if $\deg(f) = \deg(g)$.

2.3 Varieties

From here on out, unless otherwise stated, I will use the word variety to mean either an affine variety or a projective variety.

Proposition 2.9. *Every algebraic set (affine or projective) is the finite union of varieties.*

Proof. Let X be an algebraic set. If X is irreducible then we are done so suppose it is not irreducible. Then $X = X_1 \cup X_1'$ with neither equal to X. If both X_1 and X_1' are irreducible then we are done. Otherwise, suppose without loss of generality that X_1 cannot be written as the finite union of varieties. Then continuing on in this fashion, one gets a chain of sets $X \supsetneq X_1 \supsetneq X_2 \supsetneq ...$ that does not terminate. Consider the ideals of each set. By Hilbert's Nullstellensatz, we have a one to one correspondence between X_i and $I(X_i)$. In particular, $I(X_i) = I(X_{i+1})$ if and only if $X_i = X_{i+1}$. This shows us that no two of the $I(X_i)$ are equivalent. Taking the ideals gives us

$$I(X) \supsetneq I(X_1) \supsetneq I(X_2) \supsetneq ...$$

which is a non stabilizing strictly ascending chain of ideals contradicting the fact that the ring is Noetherian. Thus we get termination and so each algebraic set is a finite union of varieties. ∎

Remark 2.10. Notice that this can be extended to a unique decomposition of a variety if we insist that no irreducible variety is contained inside another in the decomposition.

Definition 2.11. Let X be a variety with X equal to the zero set of S, a set of polynomials over $k[x_1, .., x_n]$. Then, X is a *smooth variety* if at every point $p \in \mathbb{A}^n$ (respectively $\mathbb{P}^{n-1}$ if the variety is projective) $\nabla f(p) := (\frac{\partial f}{\partial x_1}(p), ..., \frac{\partial f}{\partial x_n}(p)) \neq 0$ for some $f \in S$.

Definition 2.12. One can equip $\mathbb{A}^n$ or $\mathbb{P}^n$ with a topology by making every algebraic set closed. This is called the *Zariski topology*.

Definition 2.13. Let X be a variety (a subset of either $\mathbb{A}^n$ or $\mathbb{P}^{n-1}$) and $x \in X$. A function f from X to k is *regular at* x if there is an open neighbourhood of x, say $x \in \mathcal{U} \subseteq X$, and two polynomials $p, q \in k[x_1, .., x_n]$ such that $f(y) = \frac{p(y)}{q(y)}$ for all $y \in \mathcal{U}$ and $q(y) \neq 0$. In the projective case, we also require that $\deg(p) = \deg(q)$. This function is said to be *regular on* X if it is regular at every point. The ring of regular functions is denoted by $\mathcal{O}_X(X)$. The ring of regular functions on an open subset of X is denoted by $\mathcal{O}_X(\mathcal{U})$. If the base variety is clear, then the subscript is often omitted.

Remark 2.14. Note that if f is regular on X, it may be that over different open sets, different polynomials p and q are used.

Definition 2.15. Let X be a variety and Y a subvariety of X. The *local ring of* X *along* Y, denoted by $\mathcal{O}_{Y,X}$, is the set of pairs $(\mathcal{U}, f)$ where $\mathcal{U} \subseteq X$ is open, $\mathcal{U} \cap Y \neq \emptyset$ and $f \in \mathcal{O}_X(\mathcal{U})$ under the identification that $(\mathcal{U}_1, f_1) = (\mathcal{U}_2, f_2)$ if $f_1 = f_2$ on $\mathcal{U}_1 \cap \mathcal{U}_2$. Note that this is a local ring with the maximal ideal consisting of pairs $(\mathcal{U}, f)$ such that $f\,|_{\mathcal{U}\cap Y} = 0$. This ideal is written as

$$\mathcal{M}_{Y,X} := \{f \in \mathcal{O}_{Y,X} \mid f(x) = 0 \text{ for all } x \in Y\}$$

Definition 2.16. The *function field* of a variety X is $\mathcal{O}_{X,X}$, the local ring of X along X. This is denoted by $k(X)$ and the non-zero elements by $k(X)^*$. In particular, if X is projective and $f = \frac{g}{h} \in k(X)^*$, then $\deg(g) = \deg(h)$ by (2.8) and thus $\deg(f) := \deg(g) - \deg(h) = 0$. Elements of the function field are called *rational functions*.

Remark 2.17. We could also have defined the function field of an affine variety to be the fraction field of the coordinate ring $\Gamma(X)$ (which is just $\mathcal{O}_X(X)$). In this manner, we could extend the definition to the projective case by restricting our variety to an affine open piece. This definition is well defined (that is, independent of the open affine piece we choose) as open sets are dense in the Zariski topology. In fact, this is precisely how we will define rational functions on an open set.

Definition 2.18. A map $\phi : X \to Y$ between two varieties X and Y is a morphism if it is continuous and for every open set $\mathcal{U} \subseteq Y$ and every regular function f on $\mathcal{U}$, we have that $f \circ \phi$ is regular on $\phi^{-1}(\mathcal{U})$. We say that X and Y are isomorphic if there exists two regular functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are the identity maps on X and Y respectively.

Definition 2.19. Let X be a variety. Then we define an open affine subset of X to be an open subset of X that is isomorphic to an open subset of $\mathbb{A}^n$.

Definition 2.20. Let X be an affine variety and $\mathcal{U}$ an open subset of X. Then a rational function on $\mathcal{U}$ is an element of the field of fractions of the ring of regular functions on $\mathcal{U}$. This is well defined since the ring of regular functions is indeed an integral domain (see [2, p.82]). More generally, if X is a variety, $\mathcal{U}$ is any open subset of X, and $\mathcal{V}$ is an open affine subset (which exists by [2, p. 25]) of X, then define a rational function on $\mathcal{U}$ to be an element on the fraction field of the ring of regular functions on $\mathcal{U} \cap \mathcal{V}$. This is well defined as open sets are dense in the Zariski topology.

Definition 2.21. Let X be a variety. Then the *dimension* of X is the transcendence degree of the function field $k(X)$ over k and is denoted by $\dim X$. If Y is a subvariety of X, then the *codimension* of Y in X is $\dim X - \dim Y$.

Proposition 2.22. *Let X be a variety and Y a subvariety of X. If $X \neq Y$ then* $\dim Y < \dim X$

Proof. See [3, p. 55]. ■

I would like to finish this section with an important idea from ring theory. These upcoming propositions help to give light to the local rings and their structure. Discrete valuation rings are the precise structure that allows us to connect divisors and functions in a natural manner.

Definition 2.23. A *discrete valuation ring* is a principal ideal domain that is not a field and contains a unique maximal ideal.

Note. A discrete valuation ring can also be described as a Noetherian ring that is not a field and with a unique maximal ideal that is principal.

Definition 2.24. In the case of a discrete valuation ring, if the maximal ideal is $M = (t)$, then t is a *uniformizer for M* .

Proposition 2.25. *Let R be a discrete valuation ring with maximal ideal $M = (t)$. For every $r \in R$, if $r \neq 0$ then there is a unique $n \in \mathbb{N}$ and $u \in R^*$ such that $r = ut^n$.*

Proof. If r is a unit, then use $n = 0$. Otherwise, $r \in (t)$ (since it is maximal) so there exists a $r_1 \in R$ such that $r = r_1 t$. Now $r_1 \in R$ so either r_1 is a unit in which case the process stops or $r_1 \in (t)$ and so there exists an $r_2 \in R$ such that $r_1 = r_2 t$. Continuing this process, eventually it is the case that $r_n \in R^*$ for otherwise, $(r_1) \subseteq (r_2) \subseteq \ldots$ is a chain of ideals that does not stabilize contradicting the fact that a discrete valuation is Noetherian. Thus, this process stops for some r_n. Then $r = r_1 t = r_2 t^2 = \ldots = r_n t^n$ with $r_n \in R^*$ as claimed. ■

With all of this in place, I conclude with a theorem that will be used in the next section. Its proof however required a bit more machinery and is omitted for succinctness.

Proposition 2.26. *Let X be a smooth variety and Y a subvariety of codimension 1. Then $\mathcal{O}_{Y,X}$ is a discrete valuation ring.*

Proof. See [2, p. 174]. ■

Chapter 3

Weil Divisors

Divisors are a very useful tool for studying algebraic geometry. They reveal a large amount of information about the variety in question. One can retrieve information about their zeros, poles, and structure of functions defined over the variety through the use of divisors. To begin, I start by looking at Weil divisors and how one can associate functions to divisors. I then prove some important properties of divisors as well as give a couple of examples.

Definition 3.1. Let X be an algebraic variety. A *Weil divisor* is a finite formal sum of codimension one subvarieties of X. This is denote by $D := \sum_{i=1}^{k} n_{Y_i} Y_i$ where each of the Y_i are subvarieties of X of codimension one.

There is a natural group structure on the set of all Weil Divisors of an algebraic variety where addition is defined variety wise on two divisors, that is,

$$\sum_{i=1}^{k} m_{Y_i} Y_i + \sum_{i=1}^{k} n_{Y_i} Y_i = \sum_{i=1}^{k} (m_{Y_i} + n_{Y_i}) Y_i$$

Let $\operatorname{Div}(X) = \{D | D \text{ is a Weil Divisor}\}$. This forms a free abelian group of codimension one subvarieties of X.

Definition 3.2. The *degree* of a divisor to be the sum of the n_{Y_i}'s. This is denoted by $\deg(D)$.

Definition 3.3. A divisor D is *effective* (denoted $D \geq 0$) if every term in the divisor is non-negative. That is, $n_{Y_i} \geq 0$ for all i. One can induce a partial order on divisors by setting $D_1 \geq D_2 \Leftrightarrow D_1 - D_2 \geq 0$.

Definition 3.4. The *support* of a divisor is the union of all Y_i with $n_{Y_i} \neq 0$.

Example 3.5. Let X be the zero set of f, where f is defined by $f(x,y,z) = xy - z^2$. Notice that X is a smooth projective plane curve. Divisors on X can be defined as formal sums of points of X. For example $D := 3[1:1:1] - [0:1:0]$ is a divisor on X. This divisor is not effective as the coefficient of $[0:1:0]$ is negative. Its support is $\{[1:1:1],[0:1:0]\}$

Next I define the order of a function. One can use the order of a function to help define a divisor canonically associated to functions.

Definition 3.6. Let X be a smooth algebraic variety and Y a subvariety of X of codimension 1. Recall from proposition (2.26) that $\mathcal{O}_{Y,X}$ is a discrete valuation ring. We define the valuation on this ring as follows. The order of a function $f \in \mathcal{O}_{Y,X}$, denoted $\operatorname{ord}_Y(f)$, to be the value of n from (2.25). That is, if $M = (t)$ is the maximal ideal for $\mathcal{O}_{Y,X}$ and $f = ut^n$ for some unit $u \in \mathcal{O}^*_{Y,X}$ then $\operatorname{ord}_Y(f) = n$.

Note that $\operatorname{ord}_Y : \mathcal{O}_{Y,X} \backslash \{0\} \to \mathbb{N}$ (alternatively, one could define $\operatorname{ord}_Y(0) = \infty$ and just extend the range). In particular, there is a natural extension to the fraction field $k(X)^*$ by defining the order of $\frac{f}{g} \in k(X)^*$ to be $\operatorname{ord}_Y(\frac{f}{g}) = \operatorname{ord}_Y(f) - \operatorname{ord}_Y(g)$ thus giving a function from $k(X)^*$ to $\mathbb{Z}$.

Proposition 3.7. *The order function has the following properties for all* $f, g \in k(X)^*$

(i) $\operatorname{ord}_Y(fg) = \operatorname{ord}_Y(f) + \operatorname{ord}_Y(g)$

(ii) Only finitely many Y *have the property that* $\operatorname{ord}_Y(f) \neq 0$

(iii) $\operatorname{ord}_Y(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_{Y,X}$

(iv) $\operatorname{ord}_Y(f) = 0 \Leftrightarrow f \in \mathcal{O}^*_{Y,X}$

(v) If X *is projective, then the following are equivalent*

1. $\operatorname{ord}_Y(f) \geq 0$ *for all* Y
2. $\operatorname{ord}_Y(f) = 0$ *for all* Y
3. $f \in k^*$ *(f is a non-zero constant function)*

Proof. (i) Let $f = u_1 t^{n_1}$ and $g = u_2 t^{n_2}$ from (2.25). Note that $fg = u_1 t^{n_1} u_2 t^{n_2} = (u_1 u_2) t^{n_1 + n_2}$ and so $\operatorname{ord}_Y(fg) = \operatorname{ord}_Y(f) + \operatorname{ord}_Y(g)$. ∎

(ii) Let $f = \frac{p(x_1,..,x_m)}{q(x_1,..,x_n)}$. If $\text{ord}_Y(f) \neq 0$, then $Y \subseteq X \cap V(pq)$. So either $X \subseteq V(pq)$ in which case $X \subseteq V(p)$ and f is the zero function (a contradiction) or $\dim(X \cap V(pq)) = \dim(X) - 1$. Using the decomposition of a variety (2.10) as well as proposition (2.22), we have that each of the components must be of dimension at most $\dim(X) - 1$. The uniqueness tells us that Y has to be contained in one of the finitely many irreducible components of $X \cap V(pq)$. Since Y also has dimension $\dim(X) - 1$, the proposition (2.22) tells us that Y has to be equal to one of the finitely many irreducible components. ■

(iii) If $\text{ord}_Y(f) =: n \geq 0$ then by (2.25), write $f = ut^n$ for some unit $u \in \mathcal{O}^*_{Y,X}$ and where the maximal ideal is (t). By closure of $\mathcal{O}_{Y,X}$, it must be that $f \in \mathcal{O}_{Y,X}$. If $f \in \mathcal{O}_{Y,X}$, then $f = ut^n$ by (2.25) and so $\text{ord}_Y(f) = n \geq 0$. ■

(iv) If $\text{ord}_Y(f) = 0$, then one can write by (2.25) $f = ut^0 = u$ for some unit $u \in \mathcal{O}^*_{Y,X}$. So f is a unit and thus $f \in \mathcal{O}^*_{Y,X}$. Conversely, if $f \in \mathcal{O}^*_{Y,X}$ then f is a unit so $f = ft^0$ and thus $\text{ord}_Y(f) = 0$. ■

(v) (1) $\Rightarrow$ (2) Let $f = \frac{g}{h}$. Since X is projective, $\deg(g) = \deg(h)$ by (2.16). If $\text{ord}_Y(f) \geq 0$ for all Y, then $\frac{g}{h} = f = ut^n$ by (2.25). Now, since u is a unit, one must have that $u = \frac{u_1}{u_2}$ with $\deg(u_1) = \deg(u_2)$. Cross multiplying and taking the degrees gives $\deg(g) + \deg(u_1) = \deg(h) + \deg(u_2) + n$ and thus $n = 0$. Hence $\text{ord}_Y(f) = 0$ for all Y.

(2) $\Rightarrow$ (1) Immediate.

(2) $\Rightarrow$ (3) By (iv), $f \in \mathcal{O}^*_{Y,X}$. Since this must hold for all Y, it is clear that f cannot have any zeroes or poles for if it did there would be a Y where the function f would not be a unit. Thus, f is a function without zeroes and poles and hence, $f \in k^*$.

(3) $\Rightarrow$ (2) Immediate. ■

Definition 3.8. Let X be a variety and $f \in k(X)^*$. The *divisor* of f is

$$\text{div}\,(f) = \sum_Y \text{ord}_Y(f) Y \in \text{Div}\,(X)$$

This is well defined by the previous proposition.

Definition 3.9. A divisor is said to be *principal* if it is the divisor of a function.

Proposition 3.10. *Every divisor over $\mathbb{A}^n$ is principal.*

Proof. Let $D := \sum_{i=1}^{k} n_{Y_i} Y_i \in \operatorname{Div}(\mathbb{A}^n)$ and suppose Y_i is defined by $f_i \in k[x_1, ..., x_n]$. Consider the function

$$f := \prod_{i=i}^{k} f_i^{n_i}$$

This is a well defined function with the property that $\operatorname{div}(f) = D$ as required. ■

Definition 3.11. Let X be an algebraic variety and let $D, D' \in \operatorname{Div}(X)$. Then D and D' are said to be *linearly equivalent* (denoted $D \sim D'$) if $D = D' + \operatorname{div}(f)$ for some $f \in k(X)^*$. This defines an equivalence relation on $\operatorname{Div}(X)$.

Definition 3.12. The *divisor class group of* X (denoted $\operatorname{Cl}(X)$) is the group of divisor classes modulo linear equivalence.

Example 3.13. Let $f(x, y, z) = y^2 z - x^3 - xz^2$ and suppose $X = V(f)$. Note that X is a smooth projective plane curve. Let $D_1 := 2[0 : 0 : 1] + [0 : 1 : 0]$ and $D_2 := 3[0 : 1 : 0]$. Notice that $D_1 \sim D_2$ as $\operatorname{div}(\frac{x}{z}) = 2[0 : 0 : 1] - 2[0 : 1 : 0] = D_1 - D_2$. In fact, $D_1 = \operatorname{div}(x)$ and $D_2 = \operatorname{div}(z)$.

Chapter 4

Cartier Divisors

With Weil divisors, subvarieties of codimension one were used to tell us properties about the variety. These can be defined locally as the zeroes and poles of a single function. Cartier divisors, give a way to reverse this process, that is I will define divisors using this local property and ensure that functions mesh appropriately in a global sense. Similar to Weil divisors, I will introduce the concept and its important properties. I will then describe how Weil and Cartier divisors are related as well as explain an important example called the canonical divisor.

Definition 4.1. Let X be an algebraic variety and I an indexing set. A *Cartier divisor* is an equivalence class of collection of pairs $[\{(\mathcal{U}_i, f_i)\}_{i\in I}]$ such that

(i) $\bigcup_{i\in I}\mathcal{U}_i = X$ with each $\mathcal{U}_i$ open

(ii) $f_i \in k(X)^*$

(iii) $f_i f_j^{-1} \in \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_j)^*$

(iv) The equivalence relation $\sim$ is defined by $\{(\mathcal{U}_i, f_i)\}_{i\in I} \sim \{(\mathcal{V}_j, g_j)\}_{j\in J} \Leftrightarrow f_i g_j^{-1} \in \mathcal{O}(\mathcal{U}_i \cap \mathcal{V}_j)^*$ for every $i \in I$ and $j \in J$

As done with Weil divisors, Cartier divisors form a group denoted $\mathrm{CaDiv}\,(X)$ by defining the sum of two divisors as

$$\{(\mathcal{U}_i, f_i)\}_{i\in I} + \{(\mathcal{V}_j, g_j)\}_{j\in J} = \{(\mathcal{U}_i \cap \mathcal{V}_j, f_i g_j)\}_{i\in I, j\in J}$$

Note that indeed $f_i g_j (f_k g_l)^{-1} \in \mathcal{O}((\mathcal{U}_i \cap \mathcal{V}_j) \cap (\mathcal{U}_k \cap \mathcal{V}_l))^*$ as $f_i f_k^{-1} \in \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_k)^*$ and $g_j g_l^{-1} \in \mathcal{O}(\mathcal{V}_j \cap \mathcal{V}_l)^*$ both hold by property (iii). Thus, our function is defined on the quadruple intersection and is invertible there.

Definition 4.2. The group of Cartier divisor classes modulo linear equivalence is called the *Picard group of* X and is denoted by $\mathrm{Pic}\,(X)$.

Definition 4.3. The *support* of a Cartier divisor is defined to be the union of the set of zeroes and poles of all of the f_i.

Definition 4.4. An *effective* Cartier divisor is a Cartier divisor such that every $f_i \in \mathcal{O}(\mathcal{U}_i)$. In other words, each f_i has no poles.

Definition 4.5. Let X be a variety and $f \in k(X)^*$. Then associate to f a Cartier divisor called the *principal Cartier divisor* by $\mathrm{div}\,(f) = \{(X, f)\}$

It turns out that there is a natural connection between Weil divisors and Cartier divisors in a smooth setting. Due to the nature of the proof, it is easier to justify this claim in a later chapter after developing more tools. For now I will state the claim as it ties in the previous two chapters very nicely.

Theorem 4.6. *Let X be a smooth variety. Then there exists a ϕ such that*

$$\phi : \mathrm{CaDiv}\,(X) \to \mathrm{Div}\,(X)$$

defines an isomorphism between Cartier divisors and Weil divisors. Further, this map extends naturally to an isomorphism between $\mathrm{Pic}\,(X)$ *and* $\mathrm{Cl}\,(X)$.

Chapter 5

Linear Systems

Linear systems will turn out to be a motivating factor for coming up with the Riemann-Roch theorem. This reveals another way of looking at divisors algebraically through the view point of linear algebra.

Definition 5.1. Let X be a variety and let $D \in \text{Div}(X)$. To D, we associate a vector space $L(D)$ over k by

$$L(D) = \{f \in k(X)^* | D + \text{div}(f) \geq 0\} \cup \{0\}$$

It is a matter of checking the definition to verify that $L(D)$ is indeed a k-vector space. We denote the vector space dimension of $L(D)$ by $l(D)$. Further we associate a projective variety, denote by $\mathbb{P}(L(D))$, to $L(D)$ via the following definition.

$$\mathbb{P}(L(D)) := L(D)/k^* = L(D)/\sim$$

where $\sim$ is the equivalence relation defined by $f \sim g \Leftrightarrow f = \lambda g$ where $\lambda \in k^*$.

Definition 5.2. The set of all effective divisors linearly equivalent to D is called the *complete linear system* and is denoted by $|D|$.

Remark 5.3. Consider the following map

$$\begin{aligned} \Phi : \mathbb{P}(L(D)) &\to |D| = \{D' \mid D' \geq 0, D' \sim D\} \\ f(\text{mod } k^*) &\mapsto D + \text{div}\,(f) \end{aligned}$$

Notice that this map is one to one for if $D + \text{div}\,(f) = D + \text{div}\,(g)$ then indeed $f = \lambda g$ and so $f = g$ in $\mathbb{P}(L(D))$. This map is also onto as any element of $|D|$ is $D + \text{div}\,(f)$ for some function f. Hence, this map is a bijection. Using this Φ map allows us to define a linear system as follows.

Definition 5.4. A *linear system* on a variety X is a set of effective divisors all linearly equivalent to a divisor D and parametrized by a linear subvariety (one generated by linear forms) of $\mathbb{P}(L(D))$. Being parameterized by a linear subvariety means that when we view our linear system under the Φ map (in particular its inverse), we get a linear subvariety of $\mathbb{P}(L(D))$.

Remark 5.5. It turns out that one can also define a linear system L as a subset of $|D|$ such that the following set is a k-vector subspace of $k(X)$.

$$V(L) := \{f \in k(X)^* \mid D + \operatorname{div}(f) \in L\} \cup \{0\}$$

This can be seen by examining the map Φ defined in (5.3). Using this correspondence, we note that $\mathbb{P}(V(D))$ will be a linear subvariety of $\mathbb{P}(L(D))$ if and only if L is a linear system.

Remark 5.6. Another important fact that will be used is that when X is projective, this dimension is finite and in fact $\dim |D| = l(D) - 1$ (see [3, p. 55]).

Proposition 5.7. *Let X be a variety and let $D, D' \in Div(X)$. Then,*

(i) $k \subseteq L(D)$ if and only if $D \geq 0$

(ii) If $D \leq D'$ then $L(D) \subseteq L(D')$

(iii) If $D' = D + div(g)$ (with $g \neq 0$) then the map $\theta : L(D') \to L(D)$ defined by $\theta(f) = gf$ is an isomorphism of k-vector spaces. In particular, the dimension $\ell(D)$ depends only on the class of D in $Pic(X)$.

Proof. (i) If $k \subseteq L(D)$, then taking some $c \in k$ one sees that $D + \operatorname{div}(c) \geq 0$ or in other words, $D \geq 0$. Moreover, if $D \geq 0$, then $D + \operatorname{div}(c) = D + 0 \geq 0$ for all $c \in k$. Thus $k \subseteq L(D)$. ■

(ii) Let $f \in L(D)$. This yields $D' + \operatorname{div}(f) \geq D + \operatorname{div}(f) \geq 0$ and thus $f \in L(D')$. ■

(iii) I claim that θ is a linear map. If $f, f' \in L(D')$ and $c \in k$ then

$$\theta(f + cf') = g(f + cf') = gf + cgf' = \theta(f) + c\theta(f')$$

Therefore the map is linear. It is one to one since if $f \in \ker(\theta)$, then $0 = \theta(f) = gf$ and since $g \neq 0$ it must be that $f = 0$. Moreover, this function is surjective for if

$f' \in L(D)$, then consider $f = \frac{f'}{g}$. This f lies in $L(D')$ since

$$D' + \operatorname{div}(f) = D' + \operatorname{div}(f') - \operatorname{div}(g) = D + \operatorname{div}(g) + \operatorname{div}(f') - \operatorname{div}(g) = D + \operatorname{div}(f') \geq 0 \tag{5.1}$$

where the last inequality holds since $f' \in L(D)$. Moreover, $\theta(f) = g\frac{f'}{g} = f'$ and so θ is surjective. Hence, it is an isomorphism as k-vector spaces. ■

Proposition 5.8. *Let X be a projective variety and let $D \in Div(X)$. If $\deg(D) < 0$ then $l(D) = 0$.*

Proof. Let $f \in L(D)$ and suppose that f is a non-zero function. Then $D + \operatorname{div}(f)$ is effective and by (2.16) it is the case that $\deg(\operatorname{div}(f)) = 0$ and hence that $0 \leq \deg(D + \operatorname{div}(f)) = \deg(D)$. This is a contradiction. Hence f is the zero function. Thus, $L(D) = \{0\}$ and hence $l(D) = 0$.

Chapter 6

Sheaves

Sheaves is the next step on the road to Riemann-Roch. The concept of a sheaf dates back about twenty years before its use in algebraic geometry in 1954 by Serre. I will give many examples of sheaves throughout this section and prove several of the important properties that will be needed in the proof of the Riemann-Roch theorem. I will discuss exact sequences as they will be used in the next section where cohomology is introduced.

Definition 6.1. Let X be a topological space. A *presheaf* $\mathcal{F}$ on X consists of the following:

(i) For each $\mathcal{U} \subseteq X$ with $\mathcal{U}$ open, a set $\mathcal{F}(\mathcal{U})$

(ii) For all open subsets $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U} \subseteq X$, a map

$$\rho_{\mathcal{U},\mathcal{V}} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$$

with $\rho_{\mathcal{U},\mathcal{U}} = \mathrm{id}_{\mathcal{F}(\mathcal{U})}$ and $\rho_{\mathcal{U},\mathcal{W}} = \rho_{\mathcal{V},\mathcal{W}} \circ \rho_{\mathcal{U},\mathcal{V}}$. These maps are sometimes called *restriction maps.*

If $\mathcal{F}(\mathcal{U})$ has additional structure such as being a group, ring, or module, then the presheaf is called a presheaf of groups, rings or modules respectively. In this case, the ρ maps defined above must also preserve this structure.

Definition 6.2. Let X be a topological space. A presheaf $\mathcal{F}$ is a *sheaf* if for any open $\mathcal{U} \subseteq X$ and any open cover $\bigcup_{i \in I} \mathcal{U}_i$, the following is true,

(i) Let $f, g \in \mathcal{F}(\mathcal{U})$ with $\rho_{\mathcal{U},\mathcal{U}_i}(f) = \rho_{\mathcal{U},\mathcal{U}_i}(g)$ for all $i \in I$. Then $f = g$.

(ii) Given $f_i \in \mathcal{F}(\mathcal{U}_i)$ such that $\rho_{\mathcal{U}_i,\mathcal{U}_i \cap \mathcal{U}_j}(f_i) = \rho_{\mathcal{U}_i,\mathcal{U}_i \cap \mathcal{U}_j}(f_j)$, then there exists a (unique) $f \in \mathcal{F}(\mathcal{U})$ with $\rho_{\mathcal{U},\mathcal{U}_i}(f) = f_i$ for all $i \in I$.

Definition 6.3. Let $\mathcal{F}$ be a sheaf on a topological space X. The set $\mathcal{F}(X)$ (treating X as an open subset of itself) is the set of *global sections* of $\mathcal{F}$.

Definition 6.4. Let X be a variety. An $\mathcal{O}_X$*-module* is a sheaf $\mathcal{F}$ on X such that

(i) For each open set $\mathcal{U} \subseteq X$, $\mathcal{F}(\mathcal{U})$ is an $\mathcal{O}_X$-module over the ring $\mathcal{O}_X(\mathcal{U})$.

(ii) For each pair of open sets $\mathcal{V} \subseteq \mathcal{U} \subseteq X$, the map $\rho_{\mathcal{U},\mathcal{V}} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$ is an $\mathcal{O}_X(\mathcal{U})$ module homomorphism.

Example 6.5. (i) *Direct Product of Sheaves*: Let $\mathcal{F}_\lambda$ for $\lambda \in \Lambda$ be sheaves on a topological space X. The sheaf $\mathcal{F}$ on X defined by

$$\mathcal{F}(\mathcal{U}) = \prod_{\lambda \in \Lambda} \mathcal{F}_\lambda(\mathcal{U})$$
$$\rho_{\mathcal{U},\mathcal{V}} = \prod_{\lambda \in \Lambda} \rho^\lambda_{\mathcal{U},\mathcal{V}}$$

is a sheaf called the direct product of sheaves. It is denoted by $\mathcal{F} = \prod_{\lambda \in \Lambda} \mathcal{F}_\lambda$.

For the following examples, let X be a variety equipped with the Zariski topology. Then the following, equipped with the natural restrictions $\rho_{\mathcal{U},\mathcal{V}}$, define sheaves of $\mathcal{O}_X$-modules.

(ii) *Skyscraper Sheaf*: Let $p \in X$. Define for an open set $\mathcal{U} \subseteq X$

$$\mathcal{K}_{(p)}(\mathcal{U}) = \begin{cases} k & \text{if } p \in \mathcal{U} \\ 0 & \text{if } p \notin \mathcal{U} \end{cases}$$

The sheaf $\mathcal{K}_{(p)}$ is called the skyscraper sheaf.

(iii) *Sheaf of Rational Functions*: Define

$$\mathcal{K}_X(\mathcal{U}) = \{ \text{ rational functions defined on } \mathcal{U}\}$$

The sheaf $\mathcal{K}_X$ is called the sheaf of rational functions.

(iv) *Sheaf of Regular Functions*: Define

$$\mathcal{O}_X(\mathcal{U}) = \{ \text{ regular functions defined on } \mathcal{U}\}$$

The sheaf $\mathcal{O}_X$ is called the sheaf of regular functions.

(v) *Sheaf of a Divisor*: Let $D := \{(\mathcal{U}_i, f_i) \mid i \in I\}$ be a Cartier divisor. Define as a sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$

$$\begin{aligned}\mathcal{L}_D(\mathcal{U}_i) &= \frac{1}{f_i}\mathcal{O}_X(\mathcal{U}_i)\\ &= \{f \in \mathcal{K}_X(U_i) \mid D + \operatorname{div}(f) \geq 0\}\end{aligned}$$

The equality is immediate becuase this sheaf must generate only functions f where $D + \operatorname{div}(f) \geq 0$ as it is a submodule of $\mathcal{K}_X$. Notice since $\mathcal{U}_i$ form an open cover, this determines $\mathcal{L}_D$. The second definition shows immediately that $\mathcal{L}_D$ depends only on the equivalence class of D. Also, from the second definition, we can see that $\mathcal{L}_{-D} = f_i\mathcal{O}_X(\mathcal{U}_i)$.

Definition 6.6. Let $\mathcal{F}, \mathcal{G}$ be two (pre)sheaves. A *morphism of (pre)sheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\phi_{\mathcal{U}}$ for each open $\mathcal{U} \subseteq X$ that preserves the structure of the sheaf (for example, if $\mathcal{F}(\mathcal{U})$ was a $\mathcal{O}_X$-module, then these homomorphisms are required to be $\mathcal{O}_X$-module homomorphisms) and such that the following diagram is commutative,

$$\begin{array}{ccc}\mathcal{F}(\mathcal{U}) & \xrightarrow{\phi_{\mathcal{U}}} & \mathcal{G}(\mathcal{U})\\ \downarrow \rho_{\mathcal{U},\mathcal{V}} & & \downarrow \hat{\rho}_{\mathcal{U},\mathcal{V}}\\ \mathcal{F}(\mathcal{V}) & \xrightarrow{\phi_{\mathcal{V}}} & \mathcal{G}(\mathcal{V})\end{array}$$

Definition 6.7. Two (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if there is a morphism $\{\phi_{\mathcal{U}}\}$ of (pre)sheaves such that for every open set $\mathcal{U} \subseteq X$, we have that $\phi|_{\mathcal{U}}$ is an isomorphism in the appropriate setting (for example, as topological spaces, groups, rings, or modules).

Definition 6.8. Let $\mathcal{F}$ be a sheaf on a topological space X and let $x \in X$. Define an equivalence relation $\sim$ on $\mathfrak{F} := \bigsqcup_{x\in\mathcal{U}} \mathcal{F}(\mathcal{U})$ as follows. Let $f \in \mathcal{F}(\mathcal{U})$ and $g \in \mathcal{F}(\mathcal{V})$. Then $f \sim g$ if $\rho_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(f) = \rho_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(g)$. The *stalk* at x, denoted $\mathcal{F}_x$ is

$$\mathcal{F}_x := \varinjlim_{x\in\mathcal{U}} \mathcal{F}(\mathcal{U}) := \mathfrak{F}/\sim$$

The construction above is called the *direct limit* of the $\mathcal{F}(\mathcal{U})$ over all open sets containing x. The elements of a stalk are called *germs*. Notice that if the sheaf is a sheaf of groups, rings, or modules, then the stalk is a group, ring, or module respectively.

Example 6.9. If one takes the sheaf of rings of regular functions $\mathcal{O}_X$, then the stalk at a point x is simply the ring of functions regular at x, namely $\mathcal{O}_{x,X}$.

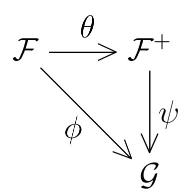
Note. Sheaf homomorphisms induce homomorphisms at the level of stalks. These will preserve the structure (say for group, ring or module homomorphisms).

Theorem 6.10. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves defined over a topological space X. Then ϕ is an isomorphism if and only if the induced maps on the stalks, denoted ϕ_x, are isomorphisms for each $x \in X$.*

Proof. The proof, while not difficult, is technical. See ([2, p. 63]).

One key idea still unexplored is the ability to turn a presheaf into a sheaf. This will become important for defining properly the tensor product of two sheaves. It turns out that there is an elegant way to perform this construction called the sheafification of a presheaf.

Definition 6.11. Let $\mathcal{F}$ be a presheaf. Then the *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^+$ together with a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that it satisfies the following universal property. For any sheaf $\mathcal{G}$ and any morphism $\phi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes:



Proposition 6.12. *The sheafification of a presheaf $\mathcal{F}$ always exists and is unique up to isomorphism.*

Proof. We construct the sheaf $\mathcal{F}^+$ as follows. Let $\mathcal{U}$ be an open set and $\mathcal{F}^+$ be the set of functions $s : \mathcal{U} \to \bigsqcup_{p \in \mathcal{U}} \mathcal{F}_p$ such that for each $p \in \mathcal{U}$, we have that $s(p) \in \mathcal{F}_p$ and there is a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ with $p \in \mathcal{V}$ and an element $t \in \mathcal{F}(\mathcal{V})$ such that for all $q \in \mathcal{V}$ the germ $t_q = s(q)$. From here, it is easy to verify that $\mathcal{F}^+$ equipped with the natural restrictions forms a sheaf. This induces a natural morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ that has the universal property. Moreover, the uniqueness is a result of the universal property. ■

Example 6.13. (i) *Inverse Image Sheaf*: Let $f : X \to Y$ be a continuous function from two topological spaces X and Y. For and sheaf $\mathcal{G}$ on Y, define the *inverse image sheaf* to be the sheafification of the presheaf

$$\mathcal{U} \mapsto \lim_{V \supseteq f(\mathcal{U})} \mathcal{G}(\mathcal{V})$$

where the limit is the usual colimit taken over any open set V that contains $f(\mathcal{U})$. This is denoted by $f^{-1}\mathcal{G}$.

(ii) *Restriction Sheaf*: Let Z be a subset of a topological space X and view Z as a topological subspace when equiped with the induced topology. If $i : Z \to X$ is the inclusion map, and $\mathcal{F}$ is a sheaf on X, then we call $i^{-1}\mathcal{F}$ the restriction of $\mathcal{F}$ to Z and denote it by $\mathcal{F} \mid_Z$.

(iii) *Image and Kernel Sheaf*: Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves defined over a topological space X. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then we define the *kernel sheaf* to be the sheaf

$$\mathcal{U} \mapsto \ker(\phi_{\mathcal{U}})$$

(note that this is a presheaf which is also a sheaf so taking the sheafification is redundant). Also, we define the *image sheaf* to be the sheafification of the presheaf

$$\mathcal{U} \mapsto \mathrm{im}(\phi_{\mathcal{U}})$$

(iv) *Homomorphism Sheaf*: Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules. For each open set $\mathcal{U} \subseteq X$, define

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(\mathcal{U}) &:= \{\mathcal{O}_X \mid_{\mathcal{U}} \text{-homomorphisms from } \mathcal{F} \mid_{\mathcal{U}} \text{ to } \mathcal{G} \mid_{\mathcal{U}}\} \\ &:= \hom_{\mathcal{O}_X|_{\mathcal{U}}}(\mathcal{F} \mid_{\mathcal{U}}, \mathcal{G} \mid_{\mathcal{U}}) \end{aligned}$$

With the natural restrictions, $\mathcal{H}om_{\mathcal{O}_X}$ defines a sheaf.

Definition 6.14. A sequence of sheaves $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is exact if $\mathrm{im}(f) = \ker(g)$.

Example 6.15. Let D be a divisor on a projective curve X (so $\dim X = 1$) and let $P \in X$. There is a natural injection from $\mathcal{L}_D$ to $\mathcal{L}_{D+P}$ directly from the definition. Next, define a map Φ from $\mathcal{L}_{D+P}$ to $\mathcal{K}_{(P)}$ as follows.

$$\Phi \mid_{\mathcal{U}} (f) = \begin{cases} 0 & \text{if } P \notin \mathcal{U} \\ f(P) & \text{if } P \in \mathcal{U} \end{cases}$$

This map is clearly onto. As such there is an exact sequence

$$0 \to \mathcal{L}_D \to \mathcal{L}_{D+P} \to K_{(P)} \to 0$$

Such exact sequences are referred to as *short exact sequences*.

Definition 6.16. Consider $\underbrace{\mathcal{O}_X \oplus ... \oplus \mathcal{O}_X}_{r} = \mathcal{O}_X^r$. This is a *free* $\mathcal{O}_X$*-module* of rank r. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Then $\mathcal{F}$ is *locally free* if for every $x \in X$, there exists an open neighbourhood $x \in \mathcal{U} \subseteq X$ such that $\mathcal{F}|_\mathcal{U} \cong \mathcal{O}_X^r|_\mathcal{U}$. A locally free sheaf of rank 1 is also called an *invertible sheaf.*

Example 6.17. Let $D = \{(\mathcal{U}_i, f_i)\}_{i \in I}$ be a Cartier divisor. Notice that $\mathcal{L}_D$ is an invertible sheaf. We use the isomorphism defined by

$$\mathcal{O}_X \mid_{\mathcal{U}_i} \to \mathcal{L}(D) \mid_{\mathcal{U}_i}$$
$$1 \mapsto f_i^{-1}$$

Notice that this can be defined for all open sets since $\mathcal{U}_i$ is an open cover so $\mathcal{U} \cap \mathcal{U}_i \neq \emptyset$ for some i. For this set $\mathcal{U}$, map 1 to f_i^{-1}. Since D is a Cartier divisor, this is well defined on the overlap.

Example 6.18. (i) *Tensor Product of Sheaves*: Define the tensor product of two $\mathcal{O}_X$-modules, $\mathcal{F}$ and $\mathcal{G}$, denoted $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, to be the sheafification of the presheaf

$$U \mapsto \mathcal{F}(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{G}(\mathcal{U})$$

When there is no confusion, the tensor product is often denoted by $\mathcal{F} \otimes \mathcal{G}$.

(ii) *Exterior Product of Sheaves*: Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Define the nth exterior product of $\mathcal{F}$ to be the sheafification of the presheaf

$$U \mapsto \bigwedge^n \mathcal{F}(\mathcal{U})$$

This along with the natural restrictions forms a presheaf whose sheafification is called the nth exterior product of two sheaves.

(iii) *Dual Sheaf*: Let $\mathcal{F}$ be a locally free sheaf of finite rank. Define $\check{\mathcal{F}} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ to be the dual sheaf of $\mathcal{F}$.

Theorem 6.19. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally free $\mathcal{O}_X$ modules of finite rank r and s respectively. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \check{\mathcal{F}} \otimes \mathcal{G}$.*

Proof. To show this isomorphism, we will use the universal property of sheafification. Let $\check{\mathcal{F}} \otimes^{\text{pre}} \mathcal{G}$ denote the presheaf of $\check{\mathcal{F}} \otimes \mathcal{G}$. By the universal property, it suffices to find an isomorphism from this presheaf to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Notice that for each $s \in \mathcal{G}(\mathcal{U})$, there is a natural map $\Psi_s : \mathcal{O}_X |_{\mathcal{U}} \to \mathcal{G} |_{\mathcal{U}}$ defined by multiplying elements by $s |_{\mathcal{U}}$. Define

$$\Phi_{\mathcal{U}} : \hom_{\mathcal{O}_X|_{\mathcal{U}}}(\mathcal{F} |_{\mathcal{U}}, \mathcal{O}_X |_{\mathcal{U}}) \otimes_{\mathcal{O}_X|_{\mathcal{U}}} \mathcal{G}(\mathcal{U}) \to \hom_{\mathcal{O}_X|_{\mathcal{U}}}(\mathcal{F} |_{\mathcal{U}}, \mathcal{G} |_{\mathcal{U}})$$
$$(\phi, s) \mapsto (\psi_s \circ \phi)$$

Now by (6.10), we have that the collection $\{\Phi_{\mathcal{U}}\}$ is an isomorphism if and only if the induced maps on the stalks are an isomorphism. As $\mathcal{F}$ is locally free, the stalks themselves are free (that is are isomorphic to $\mathcal{O}_{x,X}$). At this level, the morphisms are exactly the canonical isomorphisms of $\mathcal{O}_{x,X}$ modules. This shows that the induced morphisms are isomorphisms at the level of stalks and hence showing the original isomorphism. ■

Proposition 6.20. *Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$ module of rank 1. Then,*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \cong \mathcal{O}_X$$

Proof. Let $\Phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(\mathcal{U})$ for some small enough open set $\mathcal{U} \in X$. Note that $\Phi(f) = f\Phi(1)$, where 1 is the identity element of $\mathcal{O}_X |_{\mathcal{U}}$, and so Φ is determined by $\Phi(1)$. There are $\mathcal{F} |_{\mathcal{U}}$ possible elements all resulting in a homomorphism. However, since $\mathcal{F}$ is a locally free sheaf of rank 1, we have that $\mathcal{F} |_{\mathcal{U}} \cong \mathcal{O}_X |_{\mathcal{U}}$. Hence $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(\mathcal{U}) \cong \mathcal{F} |_{\mathcal{U}} \cong \mathcal{O}_X |_{\mathcal{U}}$. This gives the required sheaf isomorphism. ■

Corollary 6.21. *Let X be a variety. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X$*

Proof. Immediate from the previous proposition. ■

Corollary 6.22. *Let $\mathcal{F}$ be an invertible sheaf on a variety X. Then $\check{\mathcal{F}}$ is invertible and $\mathcal{F} \otimes \check{\mathcal{F}} \cong \mathcal{O}_X$.*

Proof. Immediate from (6.19) and (6.20).

Remark 6.23. Notice that the set of invertible sheaves forms a group with the operation of tensor product and $\mathcal{O}_X$ as the identity element. In particular, the sheaf satisfying $\mathcal{F} \otimes \mathcal{G} = \mathcal{O}_X$ is unique since if two sheaves say $\mathcal{G}, \mathcal{H}$ satisfy

$$\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X \cong \mathcal{F} \otimes \mathcal{H}$$

Then tensoring both sides by $\check{\mathcal{F}}$ gives us $\mathcal{G} \cong \mathcal{H}$.

Example 6.24. Let $D := \{(\mathcal{U}_i, f_i) | i \in I\}$ be a Cartier divisor. Note that $\mathcal{L}_D \otimes \mathcal{L}_{-D} \cong \mathcal{O}_X$. We use the universal property to show this. Denote the presheaf of $\mathcal{L}_D \otimes \mathcal{L}_{-D}$ by $\mathcal{L}_D \otimes^{\text{pre}} \mathcal{L}_{-D}$. Then we define a morphism $\phi_{\mathcal{U}}$ on an open set $\mathcal{U} \subseteq X$ by defining it on its intersection of the open cover via

$$(\mathcal{L}_D \otimes^{\text{pre}} \mathcal{L}_{-D})(\mathcal{U} \cap \mathcal{U}_i) \to \mathcal{O}_X(\mathcal{U} \cap \mathcal{U}_i)$$
$$f \otimes g \mapsto fg$$

This map is well defined on the overlaps by the properties of the Cartier divisor D and thus forms a map from $(\mathcal{L}_D \otimes^{\text{pre}} \mathcal{L}_{-D})(\mathcal{U})$ to $\mathcal{O}_X(\mathcal{U})$. This map defines an isomorphism of rings. It is clear that the map is onto for if $f \in \mathcal{O}_X(\mathcal{U} \cap \mathcal{U}_i)$, then $\frac{1}{f_i} \otimes f_i f \mapsto f$. This is one to one for any element of the tensor product can be described as $\frac{1}{f_i} \otimes f_i h$ by noting $\frac{1}{f_i} g \otimes f_i h = \frac{1}{f_i} \otimes f_i g h$. It is clear from the definition that $\{\phi_{\mathcal{U}}\}$ is a morphism of sheaves and hence is an isomorphism of sheaves. Note in particular that this means that $\mathcal{L}_{-D} \cong \check{\mathcal{L}}_D$ by the uniqueness discussed in the previous remark.

Theorem 6.25. *Let D_1, D_2 be Cartier divisors on a variety X. Then $\mathcal{L}_{D_1 - D_2} \cong \mathcal{L}_{D_1} \otimes \check{\mathcal{L}}_{D_2}$*

Proof. Let $D_1 = \{(\mathcal{U}_i, f_i)\}_{i \in I}$ and $D_2 = \{(\mathcal{V}_j, g_j)\}_{j \in J}$ so $D_1 - D_2 = \{(\mathcal{U}_i \cap \mathcal{V}_j, f_i g_j^{-1})\}_{i \in I, j \in J}$. By the above remark (6.23), it suffices to show that $\mathcal{L}_{D_1 - D_2} \cong \mathcal{L}_{D_1} \otimes \mathcal{L}_{-D_2}$. To show this isomorphism, we will use the universal property of sheafification. Let $\mathcal{L}_{D_1} \otimes^{\text{pre}} \mathcal{L}_{-D_2}$ denote the presheaf of $\mathcal{L}_{D_1} \otimes \mathcal{L}_{-D_2}$. By the universal property, it suffices to find an isomorphism from this presheaf to $\mathcal{L}_{D_1 - D_2}$. Define a ring homomorphism $\Phi_{\mathcal{U}}$ for any open set $\mathcal{U} \subseteq X$ by defining it on the intersections of the open cover $\mathcal{U}_i \cap V_j$ via,

$$(\mathcal{L}_{D_1} \otimes^{\text{pre}} \mathcal{L}_{-D_2})(\mathcal{U} \cap \mathcal{U}_i \cap V_j) \to \mathcal{L}_{D_1 - D_2}(\mathcal{U} \cap \mathcal{U}_i \cap V_j)$$
$$f \otimes g \mapsto fg$$

Similar to (6.24), we can see that we have defined a map on all of $\mathcal{U}$ by considering it over every set in the open cover. The function fg is in $\mathcal{L}_{D_1 - D_2}(\mathcal{U} \cap \mathcal{U}_i \cap V_j)$ since $f = \frac{1}{f_i} h_1$ and $g = \frac{1}{g_j} h_2$ and so $fg = \frac{1}{f_i g_j} h_1 h_2 \in \mathcal{L}_{D_1 - D_2}(\mathcal{U} \cap \mathcal{U}_i \cap V_j)$. On the overlaps, the maps are well defined by the properties of Cartier divisors. Just as before, we can show that this map is indeed an isomorphism at the level of rings. More importantly, $\{\Phi_{\mathcal{U}}\}$ defines an isomorphism of sheaves showing the two sheaves are isomorphic. ∎

Corollary 6.26. $\mathcal{L}_0 \cong \mathcal{O}_X$

Proof. Let D be a Cartier divisor on X. Then the above shows that

$$\mathcal{L}_0 = \mathcal{L}_{D-D} \cong \check{\mathcal{L}_D} \otimes \mathcal{L}_D \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_D, \mathcal{L}_D) \cong \mathcal{O}_X$$

where the last two equalities holds by (6.25) and since $\mathcal{L}_D$ is locally free of rank 1 (6.20) applies. ■

6.1 Correspondence Between Sheaves and Cartier Divisors

We would like to reinterpret Cartier divisors in the language of sheaves. In order to do this, we shall restrict our attention to sheaves over an abelian group. This allows us to define the quotient sheaf.

Definition 6.27. Let $\mathcal{F}$ be a sheaf. A sheaf $\mathcal{G}$ is said to be a *subsheaf* of $\mathcal{F}$ if for every open set $\mathcal{U} \subseteq X$, $\mathcal{G}(\mathcal{U})$ is a subgroup of $\mathcal{F}(\mathcal{U})$ and the restriction maps of $\mathcal{G}$ are the maps induced by $\mathcal{F}$.

Definition 6.28. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$ and let $\phi : \mathcal{G} \to \mathcal{F}$ be the natural inclusion map. Define the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ to be the sheafification of the presheaf $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})/\mathcal{G}(\mathcal{U})$.

Definition 6.29. A *Cartier divisor* on a variety X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ where the $*$ denotes the set of invertible elements of their respective sheaves.

Remark 6.30. This definition completely coincides with our previous definition of a Cartier divisor. If D is a Cartier divisor according to our previous definition, then it is clear from that definition that we get a global section of the presheaf of our quotient sheaf. Using the sheafification map gives us one for the quotient sheaf. Now, consider an element of the global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. Denote the presheaf by $\mathcal{F}$ and $\mathcal{F}^+ := \mathcal{K}_X^*/\mathcal{O}_X^*$, the sheafification of the presheaf. This element is a map $s(x) = s_x$ where $s_x \in \mathcal{F}_x$ and obeys an additional condition. This condition states that there is an open covering $\mathcal{U}_i$ of X and sections $f_i \in \mathcal{F}(\mathcal{U}_i)$ with the property that whenever $x \in \mathcal{U}_i$, then s_x is the germ of f_i at x. When we look at what $\mathcal{F}$ is, it is immediate that we have defined the conditions for the first definition of a Cartier divisor.

It turns out that there is a nice correspondence between Cartier divisors and invertible sheaves which I now describe.

Theorem 6.31. *The map $D \mapsto \mathcal{L}_D$ gives a one to one correspondence between Cartier divisors on X and invertible subsheaves on $\mathcal{K}_X$.*

Proof. Notice that $\mathcal{L}_D$ is an invertible sheaf by (6.17) and hence the map takes divisors into invertible sheaves. Next, start with an invertible subsheaf of $\mathcal{K}_X$ say $\mathcal{F}$ and take f_i on the open set $\mathcal{U}_i$ to be the inverse of a local generator of the sheaf on $\mathcal{U}_i$. Doing this for an open cover will give us a Cartier divisor say D. Applying the map gives us $\mathcal{L}_D = \frac{1}{f_i}\mathcal{O}_x(\mathcal{U}_i) = \mathcal{F}(\mathcal{U}_i)$ true by choice of f_i as the inverse of a local generator. This gives the required one to one correspondence. ■

6.2 Connection between Weil Divisors and Cartier Divisors

With the language of sheaves behind us, I can now prove the correspondence between Weil divisors and Cartier divisors on smooth varieties. Before I do this I need one more definition.

Definition 6.32. Let X be a variety and Y a subvariety. Let $\mathcal{I}_Y(\mathcal{U})$ be the ideal in the ring $\mathcal{O}_X(\mathcal{U})$ consisting of the regular functions which vanish at all points of $Y \cap \mathcal{U}$. The presheaf $\mathcal{U} \mapsto \mathcal{I}_Y(\mathcal{U})$ is a sheaf called the *sheaf of ideals* $\mathcal{I}_Y$ of Y. Notice that this is a subsheaf of $\mathcal{O}_X$. We can actually define a sheaf of ideals to be a subsheaf of $\mathcal{O}_X$, say $\mathcal{I}$, such that $\mathcal{I}(\mathcal{U})$ is an ideal of $\mathcal{O}_X(\mathcal{U})$.

Theorem 6.33. *Let X be a smooth variety. Then there exists a ϕ such that*

$$\phi : \mathrm{CaDiv}\,(X) \to \mathrm{Div}\,(X)$$

defines an isomorphism between Cartier divisors and Weil divisors. Further, this map extends naturally to an isomorphism between $\mathrm{Pic}\,(X)$ *and* $\mathrm{Cl}\,(X)$*.*

Proof. Let Y be an irreducible subvariety of codimension one of X and let $D = \{(\mathcal{U}_i, f_i)\}_{i \in I}$ be a Cartier divisor. The goal is to define a notion of order for D along Y. So pick a $\mathcal{U}_i$ so that $\mathcal{U}_i \cap Y \neq \emptyset$. Define $\mathrm{ord}_Y(D) = \mathrm{ord}_Y(f_i)$ (here, smoothness is necessary to ensure that the notion of an order is well defined). This notion is well defined for if some other $\mathcal{U}_j$ was chosen so that $\mathcal{U}_j \cap Y \neq \emptyset$, then by the definiton of a Cartier divisor we have that $f_i f_j^{-1} \in \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_j)^*$. This means that this function is invertible. So by (3.7) it is the case that $\text{ord}_Y(f_i f_j^{-1}) = 0$ and thus $\text{ord}_Y(f_i) = \text{ord}_Y(f_j)$. Now, map the Cartier divisor D to the Weil divisor $\sum \text{ord}_Y(D)$. The construction also clearly takes principal Cartier divisors to principal Weil divisors and effective Cartier divisors to effective Weil divisors.

For the other direction, let $D = \sum n_{Y_i} Y_i$ be a Weil divisor. Next, pick an $x \in X$ and consider the local ring $\mathcal{O}_{x,X}$. Notice that by 2.17 each Y_i corresponds to an ideal sheaf $\mathcal{P}_i = \mathcal{I}(Y_i)$ in $\Gamma(X)$. Then $\mathcal{P}_i \mathcal{O}_{x,X}$ is principal by [2, p.7]. So let f_x be a generator for $\mathcal{P}_i \mathcal{O}_{x,X}$. Then on some open neighbourhood $\mathcal{U}_x$ of x, it is true that $\div(f_x) = D|_{\mathcal{U}_x}$. It is clear that $D' := (\mathcal{U}_x, f_x)$ is a Cartier divisor. Moreover, it is well defined since if f_x and g_x are defined on the same open set $\mathcal{U}_x$ then $f_x g_x^{-1} \in \mathcal{O}(\mathcal{U}_x)*$ and by definition define the same Cartier divisor.

It is easy to show that this construction gives an isomorphism. Moreover, this isomorphism can be viewed in the natural way to give an isomorphism from $\text{Pic}\,(X)$ to $\text{Cl}\,(X)$ as it sends principal divisors to principal divisors and likewise for effective divisors. This gives the two isomorphisms as needed in the proof. ■

6.3 Canonical Class

Now, we can define the canonical class. The canonical class is the set of all canonical divisors of a variety. I will describe how this class is constructed in this section. First, we use machinery from module theory to describe differentials so that later we can extend these results to sheaves. To do this, we also need to describe how to create a sheaf from a module. We then describe the sheaf of relative differential forms by piecing together affine pieces and gluing them together.

For this section, let A be a commutative ring with identity, B an A-algebra, and M a B-module.

Definition 6.34. A *A-derivation* of B into M is a map $d : B \to M$ such that for all $a \in A$ and $b, b' \in B$,

(i) $d(b + b') = db + db'$

(ii) $d(bb') = bdb' + b'db$

(iii) $da = 0$

Definition 6.35. The *module of relative forms* of B over A is a B-module $\Omega_{B/A}$ together with an A-derivation $d : B \to \Omega_{B/A}$ that satisfies the following universal property. For any B-module M, and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that the following diagram commutes.

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \searrow^{d'} & \downarrow f \\ & & M \end{array}$$

With this in place, we now wish to decribe the sheaf of relative differentials. Essentially what we want is to take the previous construction and use it in the setting of sheaves.

Definition 6.36. Let X be a topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of rings with $f : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. This allows us to consider $\mathcal{G}$ as an $\mathcal{F}$-module. Define $\Omega^{\text{pre}}_{\mathcal{G}/\mathcal{F}}$ to be the presheaf defined by

$$\mathcal{U} \mapsto \Omega_{\mathcal{G}(\mathcal{U})/\mathcal{F}(\mathcal{U})}$$

with restrictions maps taken using (for $\mathcal{V} \subseteq \mathcal{U}$ an open set)

$$\mathcal{G}(\mathcal{U}) \to \mathcal{G}(\mathcal{V}) \to \Omega^{\text{pre}}_{\mathcal{G}/\mathcal{F}}(\mathcal{V})$$

where the maps are defined by the usual restrictions. Notice that this map is a $\mathcal{F}(\mathcal{U})$ derivation and so it factors through $\Omega^{\text{pre}}_{\mathcal{G}/\mathcal{F}}(\mathcal{U})$ giving the restriction maps for $\Omega^{\text{pre}}_{\mathcal{G}/\mathcal{F}}$. Now that we have a presheaf, we sheafify it to get $\Omega_{\mathcal{G}/\mathcal{F}}$, the *sheaf of relative differentials*.

Definition 6.37. Let $f : X \to Y$ be a morphism of varieties. Using the inverse image sheaf, we can retrieve a morphism

$$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

which turns $\mathcal{O}_X$ into a $f^{-1}\mathcal{O}_Y$ module. We define $\Omega_{X/Y}$ to be $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

Example 6.38. Let's discuss an important example of the sheaf of differentials. Let X be a smooth variety and for a fixed $p \in X$, consider $\Omega_{X/\{p\}}$. Notice that there is only one morphism from X to $\{p\}$, say f. In this case, notice that $f^{-1}(\mathcal{O}_{\{p\}})$ is a sheaf where all of the stalks are just k. This is true because the inverse image sheaf in this case is a colimit where there is only one non-empty open set, namely $\{p\}$ and so the presheaf construction for the inverse image sheaf simply sends an open set $\mathcal{U}$ of X to $\mathcal{O}_{\{p\}}(\{p\}) = k$. By construction, this fact still holds true for the sheafification. Hence the stalk of $f^{-1}(\mathcal{O}_{\{p\}})$ is a direct limit taken where every element of the direct limit is just k and so the stalk is simply k. Notice in particular that our result doesn't depend on $\{p\}$ nor on the morphism (as there is only one to choose).

Definition 6.39. Let X be a smooth variety. The *canonical sheaf* is $\omega_X = \bigwedge_{i=1}^{\dim(X)} \Omega_{X/\{p\}}$, the top exterior product of $\Omega_{X/\{p\}}$ for any $p \in X$. Notice that since X is smooth, our sheaf $\Omega_{X/\{p\}}$ is a locally free sheaf of rank $\dim(X)$ and hence ω_X is an invertible sheaf (see [2, p. 177]).

One of the most important properties of the canonical sheaf is that it is an invertible sheaf. So by (6.31) there is an associated divisor to ω_X. This divisor will be called a *canonical divisor* and denoted by K. Notice that a canonical divisor is unique up to linear equivalence. This definition of the canonical class uses a lot of the language of sheaves. From a different view point, we can define local parameters and local coordinates. Then, we will define the divisor associated to ω_X.

Definition 6.40. *Local Parameters.* Let x be a smooth point on a variety X with $\dim(X) = n$. Then $t_1, .., t_n \in \mathcal{O}_{x,X}$ are called local parameters at x if the $t_i \in \mathcal{M}_x$ and if they give a basis of $\mathcal{M}_x/\mathcal{M}_x^2$

Definition 6.41. *Local Coordinates.* Let x be a smooth point on a variety X with $\dim(X) = n$. Then $t_1, .., t_n \in \mathcal{O}_{x,X}$ are called local coordinates on X if $t_i - t_i(x)$ give local parameters at all x in X.

Example 6.42. Over $\mathbb{A}^n$, $x_1, .., x_n$ is a set of *local coordinates* for any point $x \in \mathbb{A}^n$.

Let's decribe the alternate viewpoint for canonical divisors. For any open set $\mathcal{U}$ (sufficiently small) choose rational functions $f_1, .., f_n$ such that they form a system of local coordinates everywhere on $\mathcal{U}$. Thus, $df_1 \wedge ... \wedge df_n \in \omega_X(\mathcal{U})$ is a nonzero differential form. Moreover, for any $\alpha \in \omega_X$, one can write $\alpha = f_{\mathcal{U}}(df_1 \wedge ... \wedge df_n)$ for some rational function $f_{\mathcal{U}}$. This is possible since the set of differential forms of degree n form a 1 dimensional linear space over $\mathcal{O}_X(\mathcal{U})$ spanned by $df_1 \wedge ... \wedge df_n$ (see [6, p. 88] for more details). Do this for an open cover of X and taking all open sets gives a divisor,

$$\operatorname{div}(\alpha) = \{(\mathcal{U}, f_{\mathcal{U}})\}$$

Note that this is well defined. If $\alpha' \in \omega_X(\mathcal{U})$ then note $\alpha' = g\alpha$ where $g \in k(X)^*$ and so

$$\operatorname{div}(\alpha') = \operatorname{div}(\alpha) + \operatorname{div}(g)$$

and thus $\alpha' \sim \alpha$ as divisors.

In either language, we get a well defined divisor class that describes the same object. In the original attempt at a result similar to the Riemann-Roch theorem, it was this piece of the puzzle that was lacking from the final formula. We conclude this section by giving an example of a computation of a canonical class.

Example 6.43. The Canonical Class of $\mathbb{P}^n$.

Let $d\frac{x_0}{x_{n+1}} \wedge d\frac{x_1}{x_{n+1}} ... \wedge d\frac{x_n}{x_{n+1}}$ be a (non-zero) differential form on $\mathbb{P}^n$. Let

$$\mathcal{U}_i = \{x \in \mathbb{P}^n | x_i = 1\}$$

Note that on $\mathcal{U}_{n+1}$ there are no zeroes or poles for the differential form as $\mathbf{1}dx_0 \wedge ... \wedge dx_n$ and the constant function $\mathbf{1}$ has no poles on the open affine set $\mathcal{U}_{n+1}$. For all other $1 \leq i \leq n$,

$$\begin{aligned}
&d\frac{x_0}{x_{n+1}} \wedge ... \wedge d\frac{x_{i-1}}{x_{n+1}} \wedge d\frac{1}{x_{n+1}} \wedge d\frac{x_{i+1}}{x_{n+1}} ... \wedge d\frac{x_n}{x_{n+1}} \\
=&(\frac{1}{x_{n+1}}dx_0 + \frac{-x_0}{x_{n+1}^2}dx_{n+1}) \wedge ... \wedge (\frac{1}{x_{n+1}}dx_{i-1} + \frac{-x_{i-1}}{x_{n+1}^2}dx_{n+1}) \wedge (\frac{-1}{x_{n+1}^2}dx_{n+1}) \\
&\quad \wedge (\frac{1}{x_{n+1}}dx_{i+1} + \frac{-x_{i+1}}{x_{n+1}^2}dx_{n+1}) ... \wedge (\frac{1}{x_{n+1}}dx_n + \frac{-x_n}{x_{n+1}^2}dx_{n+1}) \\
=&(\frac{1}{x_{n+1}^{i-1}}(dx_0 \wedge dx_1 \wedge ... \wedge dx_{i-1})) \wedge (\frac{-1}{x_{n+1}^2}dx_{n+1}) \wedge (\frac{1}{x_{n+1}^{n-i}}(dx_{i+1} \wedge ... \wedge dx_n)) \\
=&\frac{-1}{x_{n+1}^{n+1}}(dx_0 \wedge dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{n+1} \wedge dx_{i+1} \wedge .. \wedge dx_n) \\
=&\frac{(-1)^{n-i+1}}{x_{n+1}^{n+1}}(dx_0 \wedge dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_{n+1})
\end{aligned}$$

Thus, as a Cartier divisor, the canonical class of $\mathbb{P}^n$ is

$$D := \{(\mathcal{U}_i, \frac{(-1)^{n-i+1}}{x_{n+1}^{n+1}}) | 1 \leq i \leq n\} \cup \{(\mathcal{U}_{n+1}, 1)\}$$

or as a Weil divisor,

$$D := -(n+1)\{x_{n+1} = 0\}$$

■

Chapter 7

Čech Cohomology

Thus far, I have introduced many topics in algebraic geometry. Unfortunately this alone is not enough machinery to prove Riemann-Roch. In this chapter, I introduce the theory of cohomology over a topological space. As I am using cohomology mainly as a tool, I will not prove many of the results, some of which have proofs that far exceed the capacity of a paper of this size. I do give references however to the proofs should the reader be interested in pursuing the results.

Throughout this section, unless otherwise stated, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and let $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$ be an open cover for X.

Definition 7.1. The *nth cochain group* is defined by

$$C^n(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0,..,i_n) \in I^{n+1}} \mathcal{F}(\mathcal{U}_{i_0} \cap ... \cap \mathcal{U}_{i_n})$$

Each cochain group consists of elements $f_{(i_0,..,i_n)} \in \mathcal{F}(\mathcal{U}_{i_0} \cap ... \cap \mathcal{U}_{i_n})$ and addition define on this group is defined componentwise.

Definition 7.2. Define *coboundary operators* from the nth cochain group to the $(n+1)$st cochain group as follows.

$$\begin{aligned} \delta_n : & C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F}) \\ & f_{(i_0,..,i_n)} \sum_{k=0}^{n+1} (-1)^k \rho_{\mathcal{V},\mathcal{W}}(f_{(i_0,..,\hat{i_k},...,i_{n+1})}) =: g_{(i_0,..,i_{n+1})} \end{aligned}$$

where $\mathcal{V} = \mathcal{U}_{i_0} \cap ... \cap \hat{\mathcal{U}}_{i_k} \cap ... \cap \mathcal{U}_{i_{n+1}}$ and $\mathcal{W} = \mathcal{U}_{i_0} \cap ... \cap \mathcal{U}_{i_{n+1}}$.

Remark 7.3. It is often clear which δ_n is being used and hence the subscript n is often omitted.

Definition 7.4. The *nth Čech cohomology groups of an open cover* $\mathfrak{U}$ *of* X is defined as follows. First $H^0(\mathfrak{U}, \mathcal{F}) := \mathcal{F}(X)$, the group of global sections. Then define for $n \geq 1$

$$n\text{-cocycles } Z^n(\mathfrak{U}, \mathcal{F}) = \ker(\delta_n)$$
$$n\text{-coboundaries } B^n(\mathfrak{U}, \mathcal{F}) = \operatorname{im}(\delta_{n-1})$$

It can be shown that the nth coboundary group lies in the nth cocycle group (by showing that $\delta_n \circ \delta_n = 0$) and thus the quotient group can be formed, namely

$$\mathcal{H}^n(\mathfrak{U}, \mathcal{F}) := Z^n(\mathfrak{U}, \mathcal{F})/B^n(\mathfrak{U}, \mathcal{F})$$

This is called the nth Čech cohomology group of $\mathcal{F}$ with respect to an open cover $\mathfrak{U}$.

Note. In the case of the first cohomology group, one denotes elements of $Z^1(\mathfrak{U}, \mathcal{F})$ by (f_{ij}). It is clear by the definition that an element (f_{ij}) is in $Z^1(\mathfrak{U}, \mathcal{F})$ if and only if $f_{ij} = f_{ij} + f_{jk}$ for all indices i, j, k. This implies that $f_{ii} = 0$ and also that $f_{ij} = -f_{ji}$. Moreover, an element (f_{ij}) is in $B^1(\mathfrak{U}, \mathcal{F})$ provided that there is an element $(g_i) \in C^0(\mathfrak{U}, \mathcal{F})$ with $f_{ij} = g_j - g_i = \delta(g_i)$ on $\mathcal{U}_i \cap \mathcal{U}_j$.

The goal now is to extend this notion to one that is independent of the chosen open cover. It turns out that the most useful way to do this is by using a direct limit construction. In order to accomplish this, I will introduce a few more definitions.

Definition 7.5. $\mathfrak{V} = \{\mathcal{V}_j\}_{j \in J}$ is said to be *finer* than $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$ if there is a function $\tau : J \to I$ such that $\mathcal{V}_j \subseteq \mathcal{U}_{\tau(j)}$. Denote this by $\mathfrak{V} < \mathfrak{U}$.

Definition 7.6. If $\mathfrak{V} < \mathfrak{U}$, define $\tau^{\mathfrak{U}}_{\mathfrak{V}} : C^n(\mathfrak{U}, \mathcal{F}) \to C^n(\mathfrak{V}, \mathcal{F})$ by

$$\tau^{\mathfrak{U}}_{\mathfrak{V}}((f_{i_0,\ldots,i_n})) = (f_{\tau(j_0),\ldots,\tau(j_n)} \mid_{V_{j_0} \cap \ldots \cap V_{j_n}})$$

where $\tau(j_k) = i_k$ for all $k \in \{1, .., n\}$. These maps induce maps on the cohomology groups going from $H^n(\mathfrak{U}, \mathcal{F})$ to $H^n(\mathfrak{V}, \mathcal{F})$. They are independent of choice of τ and if $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ then $\tau^{\mathfrak{V}}_{\mathfrak{W}} \circ \tau^{\mathfrak{U}}_{\mathfrak{V}} = \tau^{\mathfrak{U}}_{\mathfrak{W}}$.

Definition 7.7. Define an equivalence relation $\sim$ on $\bigsqcup_{\mathfrak{U}} H^n(\mathfrak{U}, \mathcal{F})$ as follows. Let $h_1 \in H^n(\mathfrak{U}, \mathcal{F})$ and $h_2 \in H^n(\mathfrak{U}', \mathcal{F})$. Then

$$h_1 \sim h_2 \Leftrightarrow \text{there exists a } \mathfrak{V} \text{ with refinement map } \tau \text{ finer than } \mathfrak{U} \text{ and } \mathfrak{U}' \text{ such that } \tau^{\mathfrak{U}}_{\mathfrak{V}}(h_1) = \tau^{\mathfrak{U}'}_{\mathfrak{V}}(h_2)$$

Definition 7.8. The *nth Čech cohomology group of* X is then defined to be

$$H^n(X, \mathcal{F}) := \bigsqcup_{\mathfrak{U}} H^n(\mathfrak{U}, \mathcal{F}) / \sim = \varinjlim_{\mathfrak{U}} H^n(\mathfrak{U}, \mathcal{F})$$

Note that in the $n = 0$ case, $H^0(X, \mathcal{F}) := \mathcal{F}(X)$.

In the case of H^1, one can actually prove a strong result.

Theorem 7.9. *For any sheaf* $\mathcal{F}$ *on* X*,* $H^1(X, \mathcal{F}) = 0 \Leftrightarrow H^1(\mathfrak{U}, \mathcal{F}) = 0$ *for every open cover* $\mathfrak{U}$ *of* X*.*

Proof. See [4, p. 296].

Definition 7.10. A set of functions ϕ_i defined on a variety X is called an integer partition of unity for an open cover $\{\mathcal{U}_i\}$ of X if

(i) The functions only take integer values.

(ii) Every point of X lies in only finitely many of the support sets of the ϕ_i (that is, the set of points where the function is non-zero).

(iii) $\sum_i \phi_i(p) = 1$ for all $p \in X$

(iv) The support of each ϕ_i is contained in $\mathcal{U}_i$.

Lemma 7.11. *Integer valued partitions of unity always exist on a variety* X*.*

Proof. Totally order the open covering $\{\mathcal{U}_i\}$ and set

$$\phi_i(p) := \begin{cases} 1 & \text{if } p \in \mathcal{U}_i \text{ and } p \notin \bigcup_{j<i} \mathcal{U}_j \\ 0 & \text{otherwise} \end{cases}$$

These functions give the desired results. ∎

Proposition 7.12. *Let* X *be a variety and let* $\mathcal{F} = \mathcal{K}_{(p)}$ *be the skyscraper sheaf for some* $p \in X$*. Then* $H^1(X, \mathcal{F}) = 0$*.*

Proof. By (7.9), it suffices to show that $H^1(\mathfrak{U}, \mathcal{F}) = 0$ where $\mathfrak{U} := \{\mathcal{U}_i\}$ is an arbitrary open covering of X. Let ϕ_i be an integer valued partition of unity, which exists by (7.11). The goal is to show that every 1-cochain (f_{ij}) is a coboundary. Define for fixed i and j,

$$\psi_{i,j} := \begin{cases} \phi_j f_{ij} & \text{if } p \text{ lies in the support of } p \\ 0 & \text{otherwise} \end{cases}$$

Next, set $g_i := -\sum_k \psi_{i,k}$ where k ranges over the entire indexing set. Notice that

$$\begin{aligned} g_j - g_i &= -\sum_k \psi_{j,k} + \sum_k \psi_{i,k} = -\sum_k \phi_k f_{jk} + \sum_k \phi_k f_{ik} \\ &= \sum_k \phi_k (f_{ik} - f_{jk}) = \sum_k \phi_k f_{ij} = f_{ij} \end{aligned}$$

Thus, $\delta(g_i) = (f_{ij})$ is a coboundary and hence the first cohomology group is trivial. ■

7.1 Theorems of Cohomology

Here I present theorems that are important in the theory of Čech cohomology. These theorems will allow for the Euler characteristic of a long exact sequence of cohomology groups to be defined which will become one of the main ingredients of the proof of the Riemann-Roch theorem.

Theorem 7.13. *(Long Exact Sequence of Cohomology) Let X be a variety and let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of sheaves. Then, there exists morphisms making the following a long exact sequence:*

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to \\ \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to ... \end{aligned}$$

Proof. See [4, p. 299]. ■

Theorem 7.14. *(Cartan-Serre) Let X be a smooth projective variety and let $\mathcal{F}$ be a locally free sheaf. Then $H^n(X, \mathcal{F})$ is finite dimensional.*

Proof. See [5]. ■

Theorem 7.15. *(Grothendieck's Vanishing Theorem) Let X be a variety of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$.*

Proof. See [2, p. 208]. ■

Theorem 7.16. *(Serre Duality) Let X be a smooth projective space, K a canonical divisor, $r := \dim X$ and $\mathcal{F}$ a locally free sheaf. Then for each $0 \leq i \leq n$, we have that,*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{L}(K) \otimes \mathcal{F})$$

Proof. See [2, p. 244]. ■

Definition 7.17. Let X be a projective variety over a field k and let $\mathcal{F}$ be a locally free sheaf on X. The *Euler Characteristic* of $\mathcal{F}$ is defined by

$$\chi(\mathcal{F}) = \sum_{i=0} (-1)^i \dim H^i(X, \mathcal{F})$$

Notice that by Cartan-Serre, and Grothendieck's vanishing theorem that this is well defined.

Proposition 7.18. *Let X be a smooth projective variety and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be a locally free sheaf on X. Suppose that*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence. Then, $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

Proof. By (7.13), this sequence induces a long exact sequence

$$\begin{aligned} 0 &\xrightarrow{f_1} H^0(X, \mathcal{F}) \xrightarrow{f_2} H^0(X, \mathcal{G}) \xrightarrow{f_3} H^0(X, \mathcal{H}) \to \\ &\xrightarrow{f_4} H^1(X, \mathcal{F}) \xrightarrow{f_5} H^1(X, \mathcal{G}) \xrightarrow{f_6} H^1(X, \mathcal{H}) \to \dots \end{aligned}$$

This sequence terminates for some n by (7.15). Notice that $\operatorname{im}(f_1) = \{0\} = \ker(f_m)$ where $m = 3n + 1$ and that $\operatorname{im}(f_i) = \ker(f_{i+1})$ for $i \in \{1, ..., m-1\}$ holding by exactness. The rank-nullity theorem gives

$$\begin{aligned} \dim H^0(X, \mathcal{F}) &= \dim \ker(f_2) + \dim \operatorname{im}(f_2) = \dim \operatorname{im}(f_1) + \dim \ker(f_3) = \dim \ker(f_3) \\ \dim H^0(X, \mathcal{G}) &= \dim \ker(f_3) + \dim \operatorname{im}(f_3) = \dim \ker(f_3) + \dim \ker(f_4) \\ \dim H^0(X, \mathcal{H}) &= \dim \ker(f_4) + \dim \operatorname{im}(f_4) = \dim \ker(f_4) + \dim \ker(f_5) \\ &\vdots \\ \dim H^n(X, \mathcal{G}) &= \dim \ker(f_{m-2}) + \dim \operatorname{im}(f_{m-2}) = \dim \ker(f_{m-2}) + \dim \ker(f_{m-1}) \\ \dim H^n(X, \mathcal{H}) &= \dim \ker(f_{m-1}) + \dim \operatorname{im}(f_{m-1}) = \dim \ker(f_{m-1}) + \dim \ker(f_m) \\ &= \dim \ker(f_{m-1}) \end{aligned}$$

Alternately subtracting and adding rows yields

$$\begin{aligned} \dim H^0(X, \mathcal{F}) - \dim H^0(X, \mathcal{G}) + \dim H^0(X, \mathcal{H}) - \dim H^1(X, \mathcal{F}) + \dim H^1(X, \mathcal{G}) \\ - \dim H^1(X, \mathcal{H}) + \dim H^2(X, \mathcal{F}) - \dots + (-1)^n \dim H^n(X, \mathcal{F}) \\ + (-1)^{n+1} \dim H^n(X, \mathcal{G}) + (-1)^n \dim H^n(X, \mathcal{H}) = 0 \end{aligned}$$

A quick check shows that the left hand side of the above equation reduces to

$$\chi(\mathcal{F}) - \chi(\mathcal{G}) + \chi(\mathcal{H}) = 0$$

This gives the result. ■

Chapter 8

Riemann Roch Theorem

One of the fundamental problems that people were interested in was to count the number of functions with a specific number of zeros and poles. This number of course is often infinite and so mathematicians focused on computing the dimension of the space of meromorphic functions with predefined zeroes and poles. This is the value of $l(D)$ where the divisor D reflects the zeroes and poles as described in the problem. It is this problem (sometimes called the Riemann-Roch problem) that the Riemann-Roch theorem solves.

The Riemann-Roch theorem was a dual effort of both Bernhard Riemann and Gustav Roch in the mid 1800s. Riemann showed that $l(D) \geq \deg(D) - \dim H^1(X, \mathcal{O}_X) + 1$ in all cases. He could not however identify when this equality held. It was Roch that provided the error term of $l(K - D)$ thus proving an equality. This theorem will apply many of the mechanics developed in the previous chapters.

Definition 8.1. The *genus* of a smooth curve is the value $\dim H^1(X, \mathcal{O}_X)$. Denote it by g.

Theorem 8.2. *(Riemann Roch Theorem) Let X be a smooth projective curve. Then for all $D \in \mathrm{Div}(X)$,*

$$l(D) - l(K - D) = \deg(D) - g + 1$$

where K is a canonical divisor of X.

Proof. Note that $l(D) = \dim H^0(X, \mathcal{L}_D)$ and

$$l(K - D) = \dim H^0(X, \mathcal{L}(K - D)) = \dim H^0(X, \mathcal{L}(K) \otimes \check{\mathcal{L}}_D)$$

where $\check{\mathcal{L}_D}$ denotes the dual sheaf. Since X is a projective variety, the Serre Duality theorem (7.16) can be invoked which says that $H^0(X, \mathcal{L}(K) \otimes \check{\mathcal{L}_D})$ and $H^1(X, \mathcal{L}_D)$ are dual and thus

$$\dim H^0(X, \mathcal{L}(K) \otimes \check{\mathcal{L}_D}) = \dim H^1(X, \mathcal{L}_D)$$

So it suffices to show that

$$\dim H^0(X, \mathcal{L}_D) - \dim H^1(X, \mathcal{L}_D) = 1 - g + \deg(D)$$

Proceed by mathematical induction. For $D = 0$, note that $\mathcal{L}_D = \mathcal{O}_X$ (6.26). Moreover, $H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X) = k$ which holds since X is a projective variety. Thus $\dim H^0(X, \mathcal{L}_D) = 1$. Moreover, by definition,

$$\dim H^1(X, \mathcal{L}_D) = \dim H^1(X, \mathcal{O}_X) = g$$

and hence

$$\dim H^0(X, \mathcal{L}_D) - \dim H^1(X, \mathcal{L}_D) = 1 - g = 1 - g + 0 = 1 - g + \deg(D)$$

proving the base case. Next, suppose the claim is true for any divisor D and show the claim is true for $D + P$ (this is inductive as any divisor here is just a formal finite sum of points). First recall by (6.15) that

$$0 \to \mathcal{L}_D \to \mathcal{L}_{D+P} \to \mathcal{K}_{(P)} \to 0$$

is an exact sequence. By (7.13), this induces a long exact sequence

$$0 \xrightarrow{f_1} H^0(X, \mathcal{L}_D) \xrightarrow{f_2} H^0(X, \mathcal{L}_{D+P}) \xrightarrow{f_3} k \to$$
$$\ldots \xrightarrow{f_4} H^1(X, \mathcal{L}_D) \xrightarrow{f_5} H^1(X, \mathcal{L}_{D+P}) \xrightarrow{f_6} 0$$

which holds since $\dim H^0(X, \mathcal{K}_{(P)}) = \dim k = 1$ and since $H^1(X, \mathcal{K}_{(P)}) = 0$ by (7.12). Next, note that $\mathrm{im}(f_1) = \{0\} = \ker(f_6)$ and $\mathrm{im}(f_i) = \ker(f_{i+1})$ for $i \in \{1..5\}$. Using the rank-nullity theorem exactly as in (7.18),

$$\dim H^0(X, \mathcal{L}_D) - \dim H^0(X, \mathcal{L}_{D+P}) + 1 - \dim H^1(X, \mathcal{L}_D) + \dim H^1(X, \mathcal{L}_{D+P}) = 0$$

Rearranging and using the fact that $\deg(D + P) - \deg(D) = 1$ yields

$$\begin{aligned} &\dim H^0(X, \mathcal{L}_D) - \dim H^1(X, \mathcal{L}_D) - \deg(D) \\ &\qquad = \dim H^0(X, \mathcal{L}_{D+P}) - \dim H^1(X, \mathcal{L}_{D+P}) - \deg(D + P) \end{aligned}$$

The induction hypothesis says that $\dim H^0(X, \mathcal{L}_D) - \dim H^1(X, \mathcal{L}_D) = 1 - g + \deg(D)$ so plugging and rearranging yields

$$\dim H^0(X, \mathcal{L}_{D+P}) - \dim H^1(X, \mathcal{L}_{D+P}) = 1 - g + \deg(D + P)$$

This completes the induction and finishes the proof. ■

Chapter 9

Applications of Riemann Roch

Armed with the Riemann-Roch theorem I can now discuss some of the many applications of this theorem to algebraic geometry. The Riemann-Roch theorem answers many of the underlying questions in regard to linear systems and gives insight to linking topology and algebraic geometry.

Corollary 9.1. *Let C be a smooth projective curve of genus g and let K be a canonical divisor on C. Then $l(K) = g$ and $\deg(K) = 2g - 2$.*

Proof. First, apply the Riemann-Roch theorem (8.2) with the divisor $D = 0$. Notice that $l(0) = 1$ and thus $1 - l(K) = l(D) - l(K - D) = \deg(D) - g + 1 = -g + 1$. Isolating gives the first result. Next, for the second result, set $D = K$ and note

$$g - 1 = l(K) - l(0) = l(D) - l(K - D) = \deg(K) - g + 1$$

Isolating gives the result. ■

Corollary 9.2. *Let X be a smooth projective curve of genus g and let $D \in \operatorname{Div}(X)$. Then if $\deg(D) \geq 2g - 1$ then $l(D) = \deg(D) - g + 1$.*

Proof. Let K be a canonical divisor on X. Corollary (9.1), gives $\deg(K) = 2g - 2$ and $\deg(D) \geq 2g - 1$ so $\deg(K - D) < 0$. Therefore, by (5.8), $l(K - D) = 0$. Then an application of Riemann-Roch (8.2), yields $l(D) = \deg(D) - g + 1$. ■

Proposition 9.3. *If D, D' are two divisors on a variety X, then there is a well-defined map:*

$$\mu : L(D) \otimes L(D') \to L(D + D')$$
$$f \otimes f' \mapsto ff'$$

Proof. It suffices to check that if $f \in L(D)$ and $f' \in L(D')$ then $ff' \in L(D+D')$. Note that $D + \mathrm{div}(f) \geq 0$ and $D' + \mathrm{div}(f') \geq 0$. Using this information, it is easy to see that

$$D + D' + \mathrm{div}(f) + \mathrm{div}(f') \geq 0$$

Hence the map is well defined. ■

Remark 9.4. This map, in general, may not be injective nor surjective. Let $X = \mathbb{P}$ so that divisors over X are simply points. Now, take $D = 2[1:0]$ and $D' = 3[1:0]$. Then by Riemann-Roch (8.2) and since the canonical divisor of X has degree 2 by (6.43) (and hence $l(K-D) = 0$) gives $l(D) = 2$ and $l(D') = 3$ so $\dim(L(D) \otimes L(D')) = 2 \cdot 3 = 6$. However, also by Riemann Roch, $l(D+D') = l(5[1:0]) = 5$ and thus, this map cannot be injective as you have a 6 dimensional space mapping into a 5 dimensional space.

For a non-surjective example, let $X = \mathbb{P}$. Suppose D, D' are divisors such that D and D' are points (of multiplicity one each) and $D + D' = 0$. Then, $L(D') = \{0\}$ as no rational function can be added (as divisors) to D' to get something effective. So the left hand side of the map is simply the trivial vector space. However, $D + D' = 0$ and $L(0) \neq \{0\}$ thus the map cannot be surjective.

Remark 9.5. It is imperative to notice the difference between this situation and the case with sheaves. With sheaves $\mathcal{L}_D \otimes \mathcal{L}_{D'} \cong \mathcal{L}_{D+D'}$ is always true. The difference is with regard to the tensor. In the case of sheaves, the tensor is over $\mathcal{O}_X$ where in the case of the proposition, the tensor is taken over the base field k.

Corollary 9.6. *(Plücker's Formula) Let X be a smooth projective curve of degree d. Then its genus is $g = \frac{(d-1)(d-2)}{2}$.*

Proof. To solve this, I shall construct a differential form whose divisor has degree $d(d-3)$. Let $p(X,Y,Z) = 0$ be the degree d defining equation for the curve. After a potential change of coordinates, assume that $Z = 0$ intersects the curve in n distinct points $P_1, .., P_n$ and that none lie on the line $Y = 0$. Moreover, without loss of generality, assume that the function $v = \frac{Z}{Y}$ gives local parameters at each point so that $\mathrm{ord}_{P_i}(v) = 1$ for each $i \in \{1, ..., n\}$.

Let $\mathcal{U} := X \backslash \{P_1, ..., P_n\}$. In affine coordinates (at $Z = 1$), where $(x,y) = (\frac{X}{Z}, \frac{Y}{Z})$, notice that $\mathcal{U}$ is defined by $p(x,y,1) = 0$. Moreover, near each P_i by the above comments, one may take $(u,v) = (\frac{X}{Y}, \frac{Z}{Y})$ as coordinates near each P_i. So take the differential form

$$\omega := \frac{dx}{p_Y(x,y,1)}$$

where the subscript denotes the partial derivative with respect to that variable. Notice that $p(X, Y, Z) = 0$ and so,

$$p_X(X, Y, Z)dX + p_Y(X, Y, Z)dY + p_Z(X, Y, Z)dZ = 0$$

and hence on the affine piece $Z = 1$, one observes that

$$p_X(x, y, 1)dX + p_Y(x, y, 1)dY = 0$$

Note also that on $Z = 1$ gives

$$dx = d(\frac{X}{Z}) = \frac{ZdX - XdZ}{Z^2} = dX$$

Similarly, $dY = dy$ and thus, combining the above yields that

$$p_X(x, y, 1)dx + p_Y(x, y, 1)dy = 0 \Rightarrow \omega = \frac{dx}{p_Y(x, y, 1)} = -\frac{dy}{p_X(x, y, 1)}$$

valid since no P_i lies on $Y = 0$. Since X is smooth, notice that X cannot have a point so that both $p_Y(x, y, 1)$ and $p_X(x, y, 1)$ are both 0. Hence ω has no poles on $\mathcal{U}$. Moreover, since x and y are local parameters at every point of $\mathcal{U}$, one sees that ω has no zeros on $\mathcal{U}$. Hence $\text{div}\,(\omega) = 0$ on $\mathcal{U}$. Next, note that $u = \frac{x}{y}$ and $v = \frac{1}{y}$. Combining these two facts yields

$$\frac{v^{d-3}dv}{p_X(u, 1, v)} = \frac{(\frac{1}{y})^{d-3}d(\frac{1}{y})}{p_X(\frac{x}{y}, 1, \frac{1}{y})} = y^{\deg(p_X)}\frac{(\frac{1}{y})^{d-3}(\frac{-dy}{y^2})}{p_X(x, y, 1)} = \frac{-dy}{p_X(x, y, 1)} = \omega$$

This shows that $\text{ord}_{P_i}(\omega) = d - 3$ for each i. Hence

$$\text{div}\,(\omega) = (d-3)\sum_{i=1}^{d} P_i$$

$$\Rightarrow \deg(\text{div}\,(\omega)) = (d-3)\sum_{i=1}^{d} \deg(P_i) = (d-3)d$$

This culminates to reveal $\deg(K) = d(d-3)$. Now, (9.1) yields that $\deg(K) = 2g - 2$ and so combining the results reveals that

$$g = \frac{1}{2}(d^2 - 3d + 2) = \frac{(d-1)(d-2)}{2}$$

as required. ■

Corollary 9.7. *(Clifford's Theorem) Let X be a smooth projective curve of genus g and let $D \in \text{Div}\,(X)$ with K a canonical divisor on X. Suppose that both $l(D)$ and $l(K - D)$ are nonzero. Then $2l(D) \leq \deg(D) + 2$.*

Proof. By Riemann-Roch (8.2), note that $l(D) - l(K - D) = \deg(D) - g + 1$ so it suffices to show that $l(D) + l(K - D) \leq g + 1$ as then summing the two results gives the desired conclusion. Next consider the map

$$\begin{aligned} |K - D| \times |D| &\to |K| \\ (E, F) &\mapsto E + F \end{aligned}$$

It is clear that this map is onto; take any divisor G in $|K|$ then note that $(G - D, D) \mapsto G$. Thus $\dim |K - D| + \dim |D| \leq \dim |K|$. By (5.6), $\dim |D| = l(D) - 1$ for any divisor D. Thus, $l(K - D) - 1 + l(D) - 1 \leq l(K) - 1 \Rightarrow l(D) + l(K - D) \leq g + 1$. This completes the claim. ∎

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