A Short Note on Equation Solving in Proofs Courses

March 26, 2018

Teaching a mathematical proofs class offers some interesting moments for introspection. For example, let's take the following question and solution problem:

Question 1: Find all real roots of the polynomial $x^2 + 5x + 4 = 0$.

This is a question that can be found in many high school textbooks. In fact many might accompany this with the following proposed solution:

Solution 1: Factoring and using the zero divisor property of the integers gives

$$\begin{aligned} x^2 + 5x + 4 &= 0 \\ (x+4)(x+1) &= 0 \\ x+4 = 0 \quad \text{or} &\quad x+1 = 0 \\ x = -4 \quad \text{or} &\quad x = -1. \end{aligned}$$

Therefore, the solutions are $x = 4$ or $x = -1$. ∎

The proposed solution which is often displayed in textbooks seems correct but has one very subtle point hidden away that is manifested by the following incorrect solution

Incorrect Solution 1: Factoring, multiplying by x and using the zero divisor property of the integers gives

$$\begin{aligned} x^2 + 5x + 4 &= 0 \\ (x+4)(x+1) &= 0 \\ x(x+4)(x+1) &= 0 \\ x = 0 \quad \text{or} \quad x+4 = 0 \quad &\text{or} \quad x+1 = 0 \\ x = 0 \quad \text{or} \quad x = -4 \quad &\text{or} \quad x = -1. \end{aligned}$$

Therefore, the solutions are $x = 0$, $x = 4$ or $x = -1$. ∎

Any mathematician immediately realizes the mistake made above. However novices can often have difficulties reconciling these errors. Indeed, to a new learner this mistake is logical, not arithmetical. The main point is that in the first solution, all steps are reversible where as in the second solution, not all steps are reversible. While in this setting, the mistake looks artificial, as we can see by the next example, the mistake can be disguised.

Question 2: Find all real solutions to the equation $\log_3(x^2 + 2x) = 1$.

Solution 2: Taking powers of 3 on both sides, simplifying and factoring yields

$$\begin{aligned}
\log_3(x^2+2x) &= 1\\
3^{\log_3(x^2+2x)} &= 3^1\\
x^2+2x &= 3\\
x^2+2x-3 &= 0\\
(x+3)(x-1) &= 0\\
x+3=0 \quad \text{or} \quad & x-1=0\\
x=-3 \quad \text{or} \quad & x=-1.
\end{aligned}$$

Therefore, the solutions are $x=-3$ or $x=-1$. ■

A quick look at the mistake yields the fact that lines two and three are not equivalent as written - the domain of the original logarithm is $x^2+2x>0$ which is lost when we transfer to $x^2+2x=3$. There are two proposed ways to fix this issue.

(i) Make each step bidirectional.

(ii) Validate the solution and hence closing the logical loop.

Since in some sense, when solving an equation, we are starting with the conclusion of a problem, it makes sense to proceed through a solution with a mention of bidirectionality.

Corrected Solution To Question 1 (Bidirectional): Factoring and using the zero divisor property of the integers gives

$$\begin{aligned}
x^2+5x+4 &= 0\\
(x+4)(x+1) &= 0\\
x+4=0 \quad \text{or} \quad & x+1=0\\
x=-4 \quad \text{or} \quad & x=-1.
\end{aligned}$$

Notice that each step above is equivalent to the previous step. Hence, $x=4$ or $x=1$ are the only solutions. ■

Corrected Solution To Question 2 (Bidirectional): Taking powers of 3 on both sides and simplifying yields

$$\begin{aligned}
\log_3(x^2+2x) &= 1\\
3^{\log_3(x^2+2x)} &= 3^1\\
x^2+2x &= 3.
\end{aligned}$$

Now, the above lines are only equivalent if we restrict the domain of valid x values to match the domain of the logarithm in the original question. This is done by noting the original equation is defined if and only if $x^2+2x>0$. We maintain this assumption throughout. Continuing to simplify and factoring yields

$$\begin{aligned}
x^2+2x-3 &= 0\\
(x+3)(x-1) &= 0\\
x+3=0 \quad \text{or} \quad & x-1=0\\
x=-3 \quad \text{or} \quad & x=1.
\end{aligned}$$

Lastly, we check these value in the condition $x^2 + 2x > 0$ and see that $3^2 + 2(3) = 15$ and $15 > 0$ and $1^2 - 2(1) = -1$ and $-1 < 0$ thus $x = 1$ is inadmissible. Therefore, the only solution is $x = 1$. ■

You can also write this out more symbolically as follows

$$\begin{aligned} & x^2 + 5x + 4 = 0 \\ \iff & (x+4)(x+1) = 0 \\ \iff & x + 4 = 0 \quad \text{or} \quad x + 1 = 0 \\ \iff & x = -4 \quad \text{or} \quad x = -1 \\ \iff & x^2 + 5x + 4 = 0. \end{aligned}$$

This can also be fixed by "closing the loop", that is, writing a bunch of statements that imply each other. These are often done in "The Following Are Equivalent" (TFAE) type proof problems. The way we can do this here is by simply plugging in the values back into the original equation to verify the solution.

Corrected Solution To Question 1 (TFAE): Factoring and using the zero divisor property of the integers gives

$$\begin{aligned} x^2 + 5x + 4 &= 0 \\ (x+4)(x+1) &= 0 \\ x + 4 = 0 \quad \text{or} \quad x + 1 &= 0 \\ x = -4 \quad \text{or} \quad x &= -1. \end{aligned}$$

Substituting these values into the original equation shows that

$$\begin{aligned} (-1)^2 + 5(-1) + 4 &= 0 \\ (-4)^2 + 5(-4) + 4 &= 0 \end{aligned}$$

and thus both $x = -4$ and $x = -1$ are solutions. ■

Corrected Solution To Question 2 (TFAE): Taking powers of 3 on both sides, simplifying and factoring yields

$$\begin{aligned} \log_3(x^2 + 2x) &= 1 \\ 3^{\log_3(x^2+2x)} &= 3^1 \\ x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x + 3 = 0 \quad \text{or} \quad x - 1 &= 0 \\ x = -3 \quad \text{or} \quad x &= -1. \end{aligned}$$

Lastly, we check these value in the original equation and see that

$$\begin{aligned} \log_3((-3)^2 + 2(-3)) &= \log_3(3) = 1 \\ \log_3((-1)^2 + 2(-1)) &= \log_3(-1) \end{aligned}$$

and the latter is inadmissible. Therefore, the only solution is $x = -3$. ■

One can sort of see the looping nature of this type of solution:

$$
\begin{aligned}
& x^2 + 5x + 4 = 0 \\
\implies & (x+4)(x+1) = 0 \\
\implies & x + 4 = 0 \quad \text{or} \quad x + 1 = 0 \\
\implies & x = -4 \quad \text{or} \quad x = -1 \\
\implies & x^2 + 5x + 4 = 0.
\end{aligned}
$$

Some ways to help students trigger this type of reflection are:

- Validating the solution completes the loop.
- Confirmation of the solution completes the problem.
- Verification of the solution concludes the problem.
- A final confirmation that the solution is indeed a solution concludes the exercise.