ANALYSIS QUALIFYING EXAM EXPANDED SYLLABUS

1. Differentiation in $\mathbb{R}^n \to \mathbb{R}^m$

Consider the map $F : \mathbb{R}^n \to \mathbb{R}^m$ where $F(\boldsymbol{x}) = (F_1(\boldsymbol{x}), \ldots, F_m(\boldsymbol{x}))$. The **directional derivative** of F in direction $\boldsymbol{v}$ is defined by:

$$D_{\boldsymbol{v}}F(\boldsymbol{a}) := \lim_{h\to 0} \frac{F(\boldsymbol{a}+h\boldsymbol{v}) - F(\boldsymbol{a})}{h}$$

and similarly the **partial derivatives** are defined by $\frac{\partial F}{\partial x_j} := D_{\boldsymbol{e_j}}F$. The function F is **differentiable** at $\boldsymbol{a}$ if $\exists$ a linear map called the **differential** (or derivative) $DF_{\boldsymbol{a}} : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{F(\boldsymbol{a}+\boldsymbol{h}) - F(\boldsymbol{a}) - DF_{\boldsymbol{a}}(\boldsymbol{h})}{|\boldsymbol{h}|} = \boldsymbol{0}.$$

Of course, differentiability may be checked component-wise in the F_j's. There are a few pleasant expression which relate these various concepts:

(1) $D_{\boldsymbol{v}}F(\boldsymbol{a}) = \sum_{j=1}^n v_j D_j F(\boldsymbol{a})$ where $\boldsymbol{v} = (v_1, \ldots, v_n)$

(2) $DF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \ldots & \frac{\partial F_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial F_m}{\partial x_1} & \ldots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$

(3) if $f : \mathbb{R}^n \to \mathbb{R}$, then $D_{\boldsymbol{v}}f(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) \cdot \boldsymbol{v}$

Differentiability Criterion. *If the partial derivatives of a function exist and are continuous in a neighbourhood of a point, then the function is differentiable at that point.*

Chain Rule. *If $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\boldsymbol{a} \in U$ and $G : V \subset \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $F(\boldsymbol{a}) \in V$, then $G \circ F$ is differentiable at $\boldsymbol{a}$ and*

$$D(G \circ F)_{\boldsymbol{a}} = DG_{F(\boldsymbol{a})} \cdot DF_{\boldsymbol{a}}.$$

Constant Functions. *If $U \subset \mathbb{R}^n$ is an open connected set, then the differentiable mapping $F : U \to \mathbb{R}^m$ is constant if and only if $DF_{\boldsymbol{x}} = 0$ for all $\boldsymbol{x} \in U$.*

Mean Value Theorem. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable and U contains the line segment $[\boldsymbol{a}, \boldsymbol{b}]$, then $\exists$ $\boldsymbol{c} \in [\boldsymbol{a}, \boldsymbol{b}]$ such that*

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = f'(\boldsymbol{c})(\boldsymbol{b} - \boldsymbol{a}) = \nabla f(\boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{a}).$$

(Does not generalize to $F : \mathbb{R}^n \to \mathbb{R}^m$.)

Equality of Mixed Partials. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2, then $D_iD_jf = D_jD_if$.*

Taylor Expansion. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^2, then*

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) = f(\boldsymbol{x}_0) + Df_{\boldsymbol{x}_0}(\boldsymbol{h}) + \frac{1}{2}\boldsymbol{h}^T \operatorname{Hess}_{\boldsymbol{x}_0}(f)\boldsymbol{h} + R_2(\boldsymbol{x}_0, \boldsymbol{h})$$

where $\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{R_2(\boldsymbol{x}_0,\boldsymbol{h})}{|\boldsymbol{h}|^2} = 0$.

Second Derivative Test for Local Extrema. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^3 and suppose that $\boldsymbol{x}_0 \in U$ is a critical point of f.*

(1) *If* $\operatorname{Hess}_{\boldsymbol{x}_0}$ *is positive-definite, then $\boldsymbol{x}_0$ is a relative minimum of f.*
(2) *If* $\operatorname{Hess}_{\boldsymbol{x}_0}$ *is negative-definite, then $\boldsymbol{x}_0$ is a relative maximum.*

Lagrange Multipliers. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. If f has a local extremum at $\boldsymbol{x}_0 \in \{\boldsymbol{x} : g_1(\boldsymbol{x}) = \ldots = g_m(\boldsymbol{x}) = 0\}$ and the vectors $\nabla g_1(\boldsymbol{x}_0), \ldots, \nabla g_m(\boldsymbol{x}_0)$ are linearly independent, then there are real numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$\nabla f(\boldsymbol{x}_0) = \lambda_1 \nabla g(\boldsymbol{x}_0) + \ldots + \lambda_m \nabla g(\boldsymbol{x}_0)$$

2. Vector Calculus

Basic Vector Operators. *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$, $F = (F_1, F_2, F_3)$ be a vector field and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field.*

(1) *Gradient:* $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
(2) *Divergence:* $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
(3) *Curl:* $\nabla \times F(\boldsymbol{x}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$
(4) *Gradients are curl-free:* $\nabla \times (\nabla f) = \boldsymbol{0}$
(5) *Curls are divergence-free:* $\nabla \cdot (\nabla \times F) = 0$

Let $\gamma(t) = (x(t), y(t), z(t))$ be a C^1 parametrized curve for $a \le t \le b$.

Path Integral. *If $f : \gamma([a, b]) \to \mathbb{R}$ is continuous we define*

$$\int_\gamma f\, ds := \int_a^b f(\gamma(t))\|\gamma'(t)\|\, dt.$$

Line Integral. *If $\boldsymbol{F} = (F_1, F_2, F_3)$ is a continuous vector field on $\gamma([a, b])$ we define*

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_\gamma F_1\, dx + F_2\, dy + F_3\, dz$$

Fundamental Theorem of Calculus for Line Integrals.

$$\int_\gamma \nabla f \cdot d\boldsymbol{s} = f(\gamma(b)) - f(\gamma(a))$$

Let $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi(u,v) = (x,y,z)$ be a C^1 parametrization of a surface. We define the tangent vectors

$$\boldsymbol{T}_v := \frac{\partial \Phi}{\partial v} \text{ and } \boldsymbol{T}_u := \frac{\partial \Phi}{\partial u}$$

and say that a surface is **regular** wherever $\boldsymbol{N} := \boldsymbol{T}_u \times \boldsymbol{T}_v \neq \boldsymbol{0}$. As for path integrals, we can integrate a scalar $f : \Phi(D) \to \mathbb{R}$ over the surface:

$$\int_\Phi f\, dS = \iint_D f(\Phi(u,v)) \|\boldsymbol{T}_u \times \boldsymbol{T}_v\|\, du\, dv.$$

Surface Integral. *If $\boldsymbol{F}$ is a continuous vector field on $\Phi(D)$, we define*

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int_D \boldsymbol{F}(\Phi(u,v)) \cdot (\boldsymbol{T}_u \times \boldsymbol{T}_v)\, du\, dv.$$

Green's Theorem. *Let $D \subset \mathbb{R}^2$ be an oriented $2-$manifold-with-boundary and suppose ∂D is positively oriented. If $P\, dx + Q\, dy$ be a C^1 differential $1-$form in a neighbourhood of D, then*

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\partial D} P\, dx + Q\, dy.$$

Stoke's Theorem. *Let D be an oriented compact $2-$manifold-with-boundary in $\mathbb{R}^3$ and let $\boldsymbol{N}$ and $\boldsymbol{T}$ be the positively oriented unit normal and unit tangent vector fields on D and ∂D respectively. If $\boldsymbol{F}$ is a C^1 vector field on an open set containing D, then*

$$\iint_D (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{N}\, dA = \int_{\partial D} \boldsymbol{F} \cdot \boldsymbol{T}\, ds.$$

Gauss' Divergence Theorem. *Let $\boldsymbol{F}$ be a C^1 vector field defined on a neighbourhood of the compact oriented smooth $n-$manifold with boundary $V \subset \mathbb{R}^n$. Then*

$$\iiint_V \nabla \cdot \boldsymbol{F} = \iint_{\partial V} \boldsymbol{F} \cdot \boldsymbol{N}\, dA$$

where $\boldsymbol{N}$ is the unit outer normal vector field on the positively oriented ∂V.

Conservative Vector Fields. *Let $\boldsymbol{F}$ be a C^1 vector field defined in a contractible region of $\mathbb{R}^2$ or $\mathbb{R}^3$. We say $\boldsymbol{F}$ is conservative if any of the following equivalent conditions hold:*

(1) $\int_\gamma \boldsymbol{F} \cdot ds = 0$ *for every closed simple curve* γ
(2) $\boldsymbol{F} = \nabla f$ *for some function* f
(3) $dF = \nabla \times \boldsymbol{F} = 0$

3. Integration in $\mathbb{R}^n \to \mathbb{R}^m$

Change of Variable. *Let $T : D \to T(D)$ be C^1-invertible on the interior of $T(D)$, then*

$$\int_{T(D)} f(x,y)\, dx\, dy = \int_D f(u(x,y), v(x,y)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

Cylindrical: $dx\, dy\, dz = r\, dr\, d\theta\, dz$. *Spherical:* $dx\, dy\, dz = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$.

Fubini-Tonelli Theorem. *Let f be $A\times B$ measurable. If any* one *of the following three conditions hold:*

$$\int_{A\times B}|f(x,y)|\,d(x,y)<\infty,\ \int_A\left(\int_B|f(x,y)|dy\right)\,dx<\infty,\ \text{or}\ \int_B\left(\int_A|f(x,y)|dx\right)\,dy<\infty$$

then

$$\int_{A\times B}f(x,y)\,d(x,y)=\int_A\left(\int_B f(x,y)dy\right)\,dx=\int_B\left(\int_A f(x,y)dx\right)\,dy.$$

(Tonelli's Theorem states that for positive functions the iterated integrals of $|f(x,y)|$ converge/diverge together.)

Dominated Convergence Theorem. *Let f_n be a sequence of measurable functions converging pointwise on A to a function f. If $\exists\ g\geq 0$ such that $|f_n|\leq g$* and *$\int_A g<\infty$ then $\int_A f_n\to\int_A f$.*

Another criterion to guarantee the result of the Dominated Convergence Theorem is that the sequence f_n be *uniformly* convergent.

Differentiating Under the Integral Sign. *Let $f:A\times J\to\mathbb{R}$ be continuous and let $\frac{\partial}{\partial t}f(x,t)$ be uniformly continuous on $A\times J$, then*

$$\frac{\partial}{\partial t}\int_A f(x,t)\,dx=\int_A\frac{\partial}{\partial t}f(x,t)\,dx.$$

Improper Integration. *Let $f:U\subset\mathbb{R}^n\to\mathbb{R}$ be locally integrable and let U_k be an approximating sequence for U. We say that $f(x)$ is* absolutely integrable *if*

$$\int_U|f|:=\lim_{k\to\infty}\int_{U_k}|f|<\infty.$$

In this case, the following limit exists and is used as a definition of the improper integral

$$\int_U f:=\lim_{k\to\infty}\int_{U_k}f.$$

Comparison Test for Improper Integrals. *Suppose that f and g are locally integrable on U with $0\leq f\leq g$. If g is absolutely integrable on U, then so is f.*

Useful test case:

$$\int_1^\infty\frac{1}{x^p}\,dx=\begin{cases}\infty & \text{if } p\leq 1\\ \frac{1}{1-p} & \text{if } p>1\end{cases}\quad\&\quad\int_0^1\frac{1}{x^p}\,dx=\begin{cases}\infty & \text{if } p\geq 1\\ \frac{1}{1-p} & \text{if } p<1\end{cases}$$

For functions of a single variable, we can sometimes assign a value even if the function is not absolutely integrable. If f is continuous on $[a,x_0[\cup]x_0,b]$, we define the **Cauchy Principal Value:**

$$PV\int_a^b f(x)\,dx:=\lim_{\epsilon\to 0}\left(\int_a^{x_0-\epsilon}+\int_{x_0+\epsilon}^b\right)f(x)\,dx.$$

4. Fundamental Real Analysis

Ratio Test (for sequences). *If* (x_n) *is a sequence of positive real numbers with* $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$ *then* $x_n \to 0$.

Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent* $\iff$ *it is bounded. Moreover, if it is decreasing (increasing) it converges to its infimum (supremum).*

Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence.*

Cauchy Convergence Criterion. *A sequence of real numbers is convergent if and only if it is Cauchy.*

Contractive Sequence Criterion. *A sequence* (x_n) *of real numbers is contractive if*

$$|x_{n+2} - x_{n+1}| \leq K|x_{n+1} - x_n|$$

for some $0 < K < 1$. *Such a sequence is Cauchy and therefore convergent.*

Cauchy-Schwarz Inequality. *For* x, y *in an inner-product space:*

$$\|\langle x, y\rangle\|^2 \leq \langle x, x\rangle\langle y, y\rangle = \|x\|^2\|y\|^2.$$

Minkowski Inequality. *Let* $1 \leq p$ *and* $a_k, b_k \in \mathbb{C}$, *then*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}.$$

Holder Inequality. *If* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $a_k, b_k \in \mathbb{C}$, *then*

$$\sum_{k=1}^{n} |a_k \cdot b_k|^p \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}.$$

Absolute Convergence Tests. *Let* a_n *and* b_n *be sequences of positive real numbers.*

(1) ***Limit Comparison Test:***

$$\sum a_n < \infty \text{ and } \limsup_{n\to\infty} \frac{b_n}{a_n} < \infty \implies \sum b_n < \infty$$

$$\sum a_n = \infty \text{ and } \liminf_{n\to\infty} \frac{b_n}{a_n} > 0 \implies \sum b_n < \infty.$$

(2) ***Ratio Test:***

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \implies \sum a_n < \infty \text{ \& } \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \implies \sum a_n = \infty$$

(3) ***Root Test:***

$$\limsup_{n\to\infty} (a_n)^{1/n} < 1 \implies \sum a_n < \infty \text{ \& } \limsup_{n\to\infty} (a_n)^{1/n} > 1 \implies \sum a_n = \infty$$

Alternating Series Test. *If* $a_n \geq 0$, $a_{n+1} \leq a_n$ *and* $\lim_{n\to\infty} a_n = 0$, *then* $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges.*

Integral Test. *If* $f : [c,\infty[\to [0,\infty[$ *is a positive decreasing continuous function with* $f(n) = a_n$ *for all* $n \geq c$, *then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_c^{\infty} f(x)\,dx < \infty.$$

Continuous Functions. *The function* $f : X \to Y$ *between metric spaces* X *and* Y *is continuous at* $x_0 \in X$ *if one of the following conditions hold:*

(1) $\forall\, \varepsilon > 0\ \exists\ \delta > 0$ *such that* $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$
(2) $f(x_0) \in U$ *open* $\implies f^{-1}(U)$ *open (for all such* U*)*
(3) $x_n \to x \implies f(x_n) \to f(x)$ *(for all such sequences* x_n*)*

Continuous Functions on Compact Sets. *Let* X *be a compact metric space.*

(1) *If* $f : X \to \mathbb{R}$ *is continuous,* $f(X)$ *is a bounded set and* f *attains its supremum and infimum.*
(2) *If* Y *is any metric space and* $f : X \to Y$ *is continuous, then* f *is uniformly continuous.*

Taylor Expansion. *Let* $f : [a,b] \to \mathbb{R}$ *be* $n-$*times differentiable. If* $x, x_0 \in [a,b]$, *then*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

where $\frac{R_n(x)}{(x-x_0)^n} \to 0$ *as* $x \to x_0$. *There are other forms of the remainder:*

(1) *If* $f^{(n+1)}$ *exists in* $]a,b[$, *we may write* $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ *for some* $\xi \in]x_0, x[$.
(2) *If* $f^{(n+1)}$ *is integrable on* $[a,b]$, *we may write* $R_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt$.

Mean Value Theorem. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous. If* f *is differentiable in* $]a,b[$, *then there is some* $\xi \in]a,b[$ *such that*

$$f(b) - f(a) = f'(\xi)(b - a).$$

Integral Mean Value Theorem. *If* $f, g : [a,b] \to \mathbb{R}$ *where* $f(x)$ *is continuous and* $0 \leq g(x)$ *is integrable, then there is some* $\xi \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

Intermediate Value Theorem. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous. If* $f(a) < k < f(b)$, *then there is some* $\xi \in [a,b]$ *such that* $f(\xi) = k$.

Darboux's Theorem. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable. If* $f'(a) < k < f'(b)$ *then there is some* $\xi \in]a,b[$ *such that* $f'(\xi) = k$.

The Class of Riemann Integrable Functions. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions (an integrable function is necessarily bounded).*

(1) *(Lebesgue) $f(x)$ is integrable $\iff$ it is continuous almost everywhere.*
(2) *(Composition) $f(x)$ integrable & $\varphi(y)$ continuous $\implies$ $\varphi \circ f$ integrable*
(3) *(Absolute Value) $f(x)$ integrable $\iff$ $|f|(x)$ integrable*
(4) *(Product) $f(x)$ & $g(x)$ integrable $\implies$ $f(x) \cdot g(x)$ integrable*

Uniformly Continuous Functions. *Let $f : A \to \mathbb{R}$ be a function.*

(1) *f is Lipschitz $\implies$ f uniformly continuous*
(2) *f uniformly continuous $\implies$ $(f(x_n))$ is Cauchy whenever (x_n) is Cauchy in A*

Continuous Extension Theorem. *A function $f :]a, b[\to \mathbb{R}$ is uniformly continuous $\iff$ it admits a continuous extension $f : [a, b] \to \mathbb{R}$.*

Weierstrass Approximation Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there is a polynomial $p_\varepsilon(x)$ such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in [a, b]$.*

Banach Fixed Point Theorem. *Let X be a complete metric space. If $T : X \to X$ satisfies $|T(x) - T(y)| \le K|x - y|$ for some $K < 1$ (i.e. it is a contraction), then T has a unique fixed point.*

Newton's Method. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and suppose that $f(a)f(b) < 0$. If $|f'(x)| \ge m > 0$ and $|f''(x)| \le M$ for all $x \in [a, b]$, then there is a subinterval $I \subset [a, b]$ containing a root of $f(r) = 0$ such that for any $x_1 \in I$ the sequence*

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to r.

Inverse Function Theorem. *Let $f : [a, b] \to \mathbb{R}$ be strictly monotone and continuous. If f is differentiable at c and $f'(c) \neq 0$ then f^{-1} is differentiable at $f(c)$ and $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$.*

More generally, let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and let $\boldsymbol{a} \in U$. If $DF(\boldsymbol{a})$ is invertible, then F is invertible in a neighbourhood of $F(\boldsymbol{a})$.

Given $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ we say that $\boldsymbol{y} = F(\boldsymbol{x})$ solves the equation $G(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ in a neighbourhood W if $G(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \iff \boldsymbol{y} = F(\boldsymbol{x})$ whenever $(\boldsymbol{x}, \boldsymbol{y}) \in W$.

Implicit Function Theorem. *Let $G : U \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1 in a neighbourhood of $(\boldsymbol{a}, \boldsymbol{b})$ with $G(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$. If $\frac{\partial G}{\partial y}(\boldsymbol{a}, \boldsymbol{b})$ is invertible, then there is a function $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ which solves $G(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ in some neighbourhood of $(\boldsymbol{a}, \boldsymbol{b})$. Moreover, since*

$$0 = \frac{d}{d\boldsymbol{x}} G(\boldsymbol{x}, F(\boldsymbol{x})) = \frac{\partial G}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{d\boldsymbol{x}} + \frac{\partial G}{\partial \boldsymbol{y}} \frac{d}{d\boldsymbol{x}} F(\boldsymbol{x}) = \frac{\partial G}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) + \frac{\partial G}{\partial \boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) F'(\boldsymbol{x})$$

we can implicitly compute $F'(\boldsymbol{x}) = -\left[\frac{\partial G}{\partial \boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})\right]^{-1} \frac{\partial G}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})$.

5. Analysis on Functions

Weierstrass M-test. *Let $M_k \geq 0$ be a sequence of complex numbers for which $\sum M_k < \infty$. If $f_n : X \subset \mathbb{C} \to \mathbb{C}$ is a sequence of functions such that $|f_n(x)| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$, then $\sum f_n(x)$ converges absolutely and uniformly on X. In this case we can integrate and differentiate the series of functions term by term.*

Differentiability of Limit Functions. *If $f_n(x)$ is a sequence of differentiable real-valued functions defined on an open interval, then*

$$\left[f_n \xrightarrow{pointwise} f \ \&\ f_n' \xrightarrow{uniformly} g\right] \implies \left[f_n \xrightarrow{uniformly} f \ \&\ f' = g\right].$$

Counterexample: $f_n(x) = \frac{1}{n}\sin(nx)$.

Power Series. *Every power series $\sum_k a_k z^k$ has a* ***radius of convergence*** $0 \leq \rho \leq +\infty$ *for which $\sum_k a_k z^k$ converges absolutely if $|z| < \rho$ and diverges if $|z| > \rho$. Moreover, the series converges uniformly on compact subsets.*

(1) *The radius of convergence is given by the* ***Cauchy-Hadamard Formula***

$$\rho = \frac{1}{\limsup |a_k|^{1/k}}.$$

(2) *Power series can be added and multiplied. In this case the radius of convergence may shrink to the minimum of the two radii.*

(3) *If $a_0 \neq 0$ we may invert a power series (or compose it with another) formally. In this case, we just know that the radius of convergence remains strictly positive.*

(4) *Uniform convergence on compact subdisks allows power series to be differentiated and integrated term by term without affecting the radius of convergence. This yields a formula for its terms:*

$$a_k = \frac{f^{(k)}(0)}{k!}$$

which shows that a power series is uniquely determined by the function it represents.

Important Power Series. (1) $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, $|z| < 1$.

(2) $\text{Log}(1-z) = \sum_{k=0}^{\infty} -\frac{z^{k+1}}{k+1}$, $|z| < 1$

(3) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$, $|z| < \infty$

(4) $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$, $|z| < \infty$

(5) $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, $|z| < \infty$

Dini's Theorem. *If $f_n : A \to \mathbb{R}$ is a sequence of continuous functions defined on a compact metric space, then*

$$f_n \xrightarrow{pointwise} f \ \&\ f_n(x) \geq f_{n+1}(x) \implies f_n \xrightarrow{uniformly} f.$$

Arzela-Ascoli Theorem. *Let $f_n : A \to \mathbb{C}$ is a sequence of continuous functions defined on a compact metric space. If (f_n) is pointwise bounded & equicontinuous, then it is uniformly bounded and contains a uniformly convergent subsequence.*

Fourier Series. *Let $f(x)$ be defined in $]-L, L[$ and determined outside this interval by $f(x+2L) = f(x)$. The Fourier Series Expansion of $f(x)$ is by definition*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi x}{L} + b_n \cdot \sin \frac{n\pi x}{L} \right) = \sum_{-\infty}^{\infty} c_n \cdot e^{i \frac{n\pi x}{L}}$$

where for any $\xi \in \mathbb{R}$ we have

$$a_n := \frac{1}{L} \int_{\xi}^{\xi+2L} f(x) \cos \frac{n\pi x}{L}\, dx,\ b_n := \frac{1}{L} \int_{\xi}^{\xi+2L} f(x) \sin \frac{n\pi x}{L}\, dx$$

and

$$c_n := \frac{1}{2L} \int_{\xi}^{\xi+2L} f(x) e^{-i \frac{n\pi x}{L}}\, dx.$$

Dirichlet Conditions. *Let $f(x)$ be defined and single-valued except possibly at a finite number of points in $]-L, L[$ and suppose that $f(x+2L) = f(x)$.*

(1) *If $f(x)$ and $f'(x)$ are both* piecewise continuous*, then the Fourier series converges pointwise to $\frac{f(x+)+f(x-)}{2}$.*
(2) *If f is* continuous *and f' is* piecewise continuous*, then the Fourier series converges uniformly to f.*

Bessel Inequality and Parseval Identity. *If $\{\varphi_k\}_{k=0}^{\infty}$ be an orthonormal system in an inner-product space V, then:*

(1) $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2$ *converges and* $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2 \leq \langle f, f \rangle = ||f||^2$ *for all $f \in V$*
(2) $\{\varphi_k\}_{k=0}^{\infty}$ *is complete* $\iff$ $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2 = \langle f, f \rangle$ *for all $f \in V$.*

6. Basic Properties of Analytic Functions

For $z \neq 0$ we have the following **multivalued functions**:

(1) $\log z := \log |z| + i \arg z = \log |z| + i \operatorname{Arg} z + 2\pi m i$, $m \in \mathbb{Z}$ and $-\pi < \operatorname{Arg} z \leq \pi$
(2) $z^{\alpha} := e^{\alpha \log z} = |z|^{\alpha} e^{i\alpha \arg z}$

along with their **principal branches** (other branches specified by choice of $m \in \mathbb{Z}$)

(1) $\operatorname{Log} z := \log |z| + i \operatorname{Arg} z$ with branch cut $z \notin]-\infty, 0]$
(2) $z^{\alpha} = |z|^{\alpha} e^{i\alpha \operatorname{Arg} z}$ with branch cut $z \notin [0, \infty[$.

Warning $\log z^n \neq n \log z$ while $\log z^{\frac{1}{n}} = \frac{1}{n} \log z$ as multivalued sets

Phase Change Lemma. *Let $g(z)$ be a single valued function, continuous in a neighbourhood of z_0. For any branch of $(z - z_0)^{\alpha}$, the function $f(z) = (z - z_0)^{\alpha} g(z)$ is multiplied by the phase factor $e^{2\pi i \alpha}$ when z traverses a complete circle about z_0 in the positive direction.*

Cauchy-Riemann Equations. *The function $f = u + iv$ defined on the domain $\mathscr{D}$ is analytic if and only if the following* two *conditions hold:*

(1) *The functions $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives.*

(2) *The Cauchy-Riemann equations hold:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Analyticity of Limit Functions. *Let $f_k(z)$ be analytic on $\mathscr{D}$.*

(1) *If $(f_k(z)) \xrightarrow{uniformly} f(z)$, then $f(z)$ is analytic on $\mathscr{D}$.*
(2) *If $(f_k(z)) \xrightarrow{normally} f(z)$, then $(f_k^{(m)}(z)) \xrightarrow{normally} f^{(m)}(z)$.*

*Here a sequence $(f_k(z))$ of analytic functions on $\mathscr{D}$ **converges normally** to the analytic function $f(z)$ if it converges uniformly to $f(z)$ on every closed disk contained in $\mathscr{D}$.*

Proof. Morera's Theorem. □

Laurent Decomposition. *Let $f(z)$ is analytic on the annulus $0 \le \rho < |z - z_0| < \sigma \le \infty$.*

(1) *The function can be decomposed as a sum $f(z) = f_\sigma(z) + f_\rho(z)$ where $f_\sigma(z)$ is analytic for $|z - z_0| < \sigma$ and $f_\rho(z)$ is analytic for $|z - z_0| > \rho$ (including at ∞). If we normalize the decomposition so that $f_\rho(\infty) = 0$, then the decomposition is unique.*
(2) *Expressing f_σ and f_ρ in terms of power series, we obtain the **Laurent series***

$$f(z) = \sum_{k=-\infty}^{-1} a_k(z - z_0)^k + \sum_{k=0}^{\infty} a_k(z - z_0)^k = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k,\ \rho < |z - z_0| < \sigma$$

converging absolutely (and uniformly on compact sub-annuli) with coefficients

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}}\, dz.$$

Proof. Uniqueness = Liouville's Theorem, existence = the Cauchy Integral Formula. □

Classification of Isolated Singularities. *Let z_0 be an isolated singularity of*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k,\ 0 < |z - z_0| < \sigma.$$

(1) *The singularity is **removable** if $a_k = 0$ for all $k < 0$.*
(2) *The singularity is a **pole** of order N if $a_{-N} \neq 0$ and $a_k = 0$ for all $k < -N$.*
(3) *Otherwise, the singularity is **essential**.*

If $z_0 = \infty$ we apply the above definitions to $f(1/z)$.

Riemann's Theorem on Removable Singularities.

isolated singularity z_0 of $f(z)$ removable $\iff$ $f(z)$ bounded in neighbourhood of z_0

Detecting Poles. *Let z_0 be an isolated singularity of $f(z)$.*

(1) *z_0 is a pole $\iff$ $|f(z)| \to \infty$ as $z \to \infty$*
(2) *z_0 is a pole of order N $\iff$ $f(z) = g(z)/(z - z_0)^N$ where $g(z)$ analytic at z_0 and $g(z_0) \neq 0$*

Casorati-Weierstrass Theorem. *If z_0 is an essential singularity of $f(z)$, then for every $w_0 \in \mathbb{C}$ there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.*

Picard's Theorem. *There are two:*

(1) ***Little***: *The image of an entire & non-constant function misses at most one point of $\mathbb{C}$.*
(2) ***Big***: *If an entire function has an essential singularity at z_0, then it assumes all possible complex values with at most a single exception infinitely often in any neighbourhood of z_0.*

Partial Fractions Decompositon. *A meromorphic function on $\mathbb{C}^*$ is rational. Every rational function has a partial fraction decomposition expressing it as the sum of a polynomial in z and its principal parts at each of its poles in the finite complex plane.*

Uniqueness Principle. *Let $f(z)$ and $g(z)$ be analytic on $\mathscr{D}$.*

(1) *If $f(z)$ is not identically zero on $\mathscr{D}$, then the zeroes of $f(z)$ are isolated.*
(2) *If $f(z) = g(z)$ for all z belonging to a subset D of $\mathscr{D}$ containing a non-isolated point, then $f(z) = g(z)$ on $\mathscr{D}$.*

Analytic Continuation. *Let $f(z) = \sum a_k(z - z_0)^k$ for $|z - z_0| < \rho$. We say $f(z)$ has an **analytic continuation** along a path $\gamma(t)$ parametrized by $a \le t \le b$ if for every t there is a convergent power series*

$$f_t(z) := \sum_{k=0}^{\infty} a_k(t)(z - \gamma(t))^n, \text{ for } |z - \gamma(t)| < \rho(t)$$

with $f_a(z) = f(z)$ and $f_s(z) = f_t(z)$ wherever their disks of convergence intersect.

*Such an analytic continuation is unique. Further, the coefficients $a_k(t)$ and radii of convergence $\rho(t)$ depend continuously on t. The **Monodromy Theorem** ensures that analytic continuations along homotopic paths upon which $f(z)$ admits analytic continuations must coincide.*

7. Harmonic Functions and Bounds on Analytic Functions

A function $u(x, y)$ is harmonic if $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the partial derivatives are continuous. If $f = u + iv$ is analytic, then $u(x, y)$ and $v(x, y)$ are harmonic as a consequence of the Cauchy-Riemann equations.

Harmonic Conjugate. *Let $\mathscr{D}$ be a simply connected domain. If $u(x, y)$ is a real valued harmonic function on $\mathscr{D}$, then there is a unique (up to adding a constant) harmonic conjugate $v(x, y)$ such that $f = u + iv$ is analytic on $\mathscr{D}$.*

Mean Value Property. *Let $u(z)$ be harmonic on $\mathscr{D}$. If $\{|z - z_0| < \rho\} \subset \mathscr{D}$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \text{ for every } 0 < r < \rho.$$

It turns out that a continuous function on $\mathscr{D}$ has the mean value property if and only if it is harmonic.

Strict Maximum Principle. *Let $u(z)$ be a real valued harmonic function on $\mathscr{D}$ with $u(z) \leq M$ for all $z \in \mathscr{D}$. If $u(z_0) = M$ for some $z_0 \in \mathscr{D}$, then $u(z) \equiv M$.*

Proof. Mean value property $\implies \{z : u(z) = M\}$ open. Continuity $\implies \{z : u(z) < M\}$ is open. Contradicts connectedness of $\mathscr{D}$. □

Maximum Modulus Principle. *Let $h(z)$ be a a complex-valued harmonic (analytic) function on $\mathscr{D}$.*

(1) *If $|h(z)| \leq M$ for all $z \in \mathscr{D}$ and $|h(z_0)| = M$ for some $z_0 \in \mathscr{D}$, then $h(z)$ is constant on $\mathscr{D}$.*
(2) *Suppose further that $\mathscr{D}$ is bounded and $h(z)$ extends continuously to $\partial\mathscr{D}$. If $|h(z)| \leq M$ for all $z \in \partial\mathscr{D}$, then $|h(z)| \leq M$ for all $z \in \mathscr{D}$.*

Cauchy Estimates. *Let $f(z)$ be analytic for $|z - z_0| \leq \rho$. If $|f(z)| \leq M$ $|z - z_0| = \rho$, then fore every $m \geq 0$ we have the following bound:*

$$|f^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M.$$

Liouville's Theorem. *A bounded entire function is constant. Here, an **entire function** is one who is analytic on the entire complex plane.*

8. Complex Integration

Fundamental Theorem of Calculus. *As usual, there are two parts:*

(1) *Let $f(z)$ be continuous on $\mathscr{D}$. If $F(z)$ is a primitive of $f(z)$, then*

$$\int_A^B f(z)\, dz = F(B) - F(A)$$

and the integral can be taken over any path in $\mathscr{D}$ from A to B.
(2) *Let $\mathscr{D}$ be simply connected. If $f(z)$ is analytic on $\mathscr{D}$, then a primitive is given by*

$$F(z) := \int_{z_0}^{z} f(w)\, dw, \text{ for } z \in \mathscr{D}$$

where z_0 is any fixed point and the integral can be taken along any path in $\mathscr{D}$.

Cauchy's Theorem. *Let $\mathscr{D}$ be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on $\mathscr{D}$ and extends smoothly to $\partial\mathscr{D}$, then $\int_{\partial\mathscr{D}} f(z)\, dz = 0$.*

Proof. A $\mathcal{C}^1$ $f(z)$ on $\mathscr{D}$ is analytic if and only if the differential $f(z)\, dz$ is closed + Green's Theorem. □

Cauchy Integral Formula. *Let $\mathscr{D}$ be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on $\mathscr{D}$ which extends smoothly to $\partial\mathscr{D}$, then for every $z_0 \in \mathscr{D}$ and $m \geq 0$ we have:*

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{\partial\mathscr{D}} \frac{f(z)}{(z - z_0)^{m+1}}\, dz$$

Morera's Theorem. *Let $f(z)$ be a* continuous *function on $\mathscr{D}$. If $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle T contained in $\mathscr{D}$, then $f(z)$ is analytic on $\mathscr{D}$.*

Residue Theorem. *Let $\mathscr{D}$ be a bounded domain with piecewise smooth $\partial\mathscr{D}$. If $f(z)$ is analytic on $\mathscr{D} \cup \partial\mathscr{D}$ except at the isolated singularities $z_1, \ldots, z_m \in \mathscr{D}$, then*

$$\int_{\partial\mathscr{D}} f(z)\, dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j].$$

If $\mathscr{D}$ is an exterior domain with piecewise smooth $\partial\mathscr{D}$ and we let a_{-1} be the coefficient of $\frac{1}{z}$ in the Laurent expansion of $f(z)$ convergent for $|z| > R$, then

$$\int_{\partial\mathscr{D}} f(z)\, dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j].$$

Equivalently, $-a_{-1} = \operatorname{Res}[f(z), \infty] = \operatorname{Res}[-\frac{1}{z^2} f(\frac{1}{z}), 0]$.

Computing Residues. *Two tricks:*

(1) *If $f(z)$ has a **pole of order n** at z_0, then*

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)].$$

(2) *If $f(z)$ and $g(z)$ are analytic at z_0 and $g(z)$ has a simple zero at z_0, then*

$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}.$$

Jordan's Lemma. *If Γ_R is the contour $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$ and $a > 0$, then*

$$\int_{\Gamma_R} |e^{iaz}||dz| < \frac{\pi}{a}.$$

Fractional Residue Theorem. *Let z_0 be a simple pole of $f(z)$ and let C_ϵ be an arc of the circle $|z - z_0| = \epsilon\}$ of angle α, then*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\, dz = \alpha i \cdot \operatorname{Res}[f(z), z_0].$$

Argument Principle. *Let $\mathscr{D}$ be bounded with piecewise smooth $\partial\mathscr{D}$. If $f(z)$ is meromorphic on $\mathscr{D}$ and extends to be analytic on $\partial\mathscr{D}$ with $f(z) \neq 0$ for all $z \in \partial\mathscr{D}$, then*

$$\int_{\partial\mathscr{D}} d \log f(z) = \int_{\partial\mathscr{D}} \frac{f'(z)}{f(z)}\, dz = 2\pi i \cdot [N_0 - N_\infty].$$

Splitting up $\log f(z) = \log|f(z)| + i\arg f(z)$ *we obtain the following restatement*

$$\int_{\partial\mathscr{D}} d\arg f(z) = 2\pi\cdot[N_0 - N_\infty].$$

Rouché's Theorem. *Let* $\mathscr{D}$ *be bounded with piecewise smooth* $\partial\mathscr{D}$. *If* $f(z)$ *and* $h(z)$ *are analytic on* $\mathscr{D}\cup\partial\mathscr{D}$ *with* $|h(z)| < |f(z)|$ *on* $\partial\mathscr{D}$, *then* $f(z)$ *and* $f(z)+h(z)$ *have the same number of zeroes in* $\mathscr{D}$ *(with multiplicities).*

Open Mapping Theorem. *A non-constant analytic map on a domain* $\mathscr{D}$ *is open.*

Fourier Transform. *If* f *be absolutely integrable on* $\mathbb{R}$, *its Fourier Transform is well defined by*

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixw}\,dx.$$

If $\hat{f}$ *is absolutely integrable and* f *is continuous at* x, *then the Fourier Inversion Formula holds:*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{ixw}\,dw.$$

Riemann-Lebesgue Lemma.

$$f\in L^1 \implies \lim_{n\to\infty}\int f(x)e^{inx}\,dx = 0.$$

9. Complex Mappings

Inverse Function Theorem. *If* $f(z)$ *is analytic on* $\mathscr{D}$ *and* $f'(z_0)\neq 0$, *then* f *has an analytic inverse* f^{-1} *in some neighbourhood of* $f(z_0)$ *where* $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$.

Conformal Mapping Theorem.

$$\left[f(z) \text{ analytic at } z_0 \;\&\; f'(z_0)\neq 0\right] \implies f(z) \text{ conformal at } z_0$$

Schwarz Lemma. *Let* $f(z)$ *be analytic for* $|z|<1$. *If* $|f(z)|\le 1$ *for all* $|z|<1$ *and* $f(0)=0$, *then:*

(1) $|f(z)|\le|z|$ *for* $|z|<1$ *where equality at any* $z_0\neq 0$ *implies that* $f(z)=e^{i\theta}z$
(2) $|f'(0)|\le 1$ *where equality implies that* $f(z)=e^{i\theta}z$.

Mobius Transformations. *A Mobius Transformation is a function of the form*

$$f(z) = \frac{az+b}{cz+d} \text{ where } ad-bc\neq 0$$

(1) *Given distinct* z_0, z_1, z_2 *in* $\mathbb{C}^*$ *there is a Mobius transformation mapping them onto any distinct* $w_0, w_1, w_2\in\mathbb{C}^*$.
(2) *The group of Mobius transformation is generated by dilations, translations and inversions.*
(3) *A Mobius transformation maps circles to circles in* $\mathbb{C}^*$.

Important Conformal Maps.

$$\mathbb{H} = \{\Im(z) > 0\},\ \mathbb{D} = \{|z| < 1\} \textit{ and } \mathbb{S}_\alpha = \{0 < \arg z < \alpha\}$$

(1) $\operatorname{Aut}\mathbb{D} = \{e^{i\theta}\frac{z-a}{1-\bar{a}z} : |a| < 1\}$
(2) $z \mapsto \frac{z-i}{z+i}$ *maps* $\mathbb{H}$ *conformally onto* $\mathbb{D}$
(3) $w \mapsto i \cdot \frac{1+w}{1-w}$ *maps* $\mathbb{D}$ *conformally onto* $\mathbb{H}$
(4) $z \mapsto z^{\frac{\pi}{\alpha}}$ *maps* $\mathbb{S}_\alpha$ *onto* $\mathbb{H}$
(5) $z \mapsto \frac{(z-z_0)}{(z-z_2)}\frac{(z_1-z_2)}{(z_1-z_0)}$ *sends* $z_0 \mapsto 0,\ z_1 \mapsto 1$ *and* $z_2 \mapsto \infty$