1. Differentiation in $\mathbb{R}^n \to \mathbb{R}^m$

Consider the map $F : \mathbb{R}^n \to \mathbb{R}^m$ where $F(x) = (F_1(x), \ldots, F_m(x))$. The directional derivative of $F$ in direction $v$ is defined by:

$$D_v F(a) := \lim_{h \to 0} \frac{F(a + hv) - F(a)}{h}$$

and similarly the partial derivatives are defined by $\frac{\partial F}{\partial x_j} := D_{e_j} F$. The function $F$ is differentiable at $a$ if $\exists$ a linear map called the differential (or derivative) $DF_a : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h \to 0} \frac{F(a + h) - F(a) - DF_a(h)}{|h|} = 0.$$

Of course, differentiability may be checked component-wise in the $F_j$’s. There are a few pleasant expression which relate these various concepts:

1. $D_v F(a) = \sum_{j=1}^n v_j D_j F(a)$ where $v = (v_1, \ldots, v_n)$
2. $DF = \left( \begin{array}{ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{array} \right)$
3. if $f : \mathbb{R}^n \to \mathbb{R}$, then $D_v f(a) = \nabla f(a) \cdot v$

Differentiability Criterion. If the partial derivatives of a function exist and are continuous in a neighbourhood of a point, then the function is differentiable at that point.

Chain Rule. If $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in U$ and $G : V \subset \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $F(a) \in V$, then $G \circ F$ is differentiable at $a$ and

$$D(G \circ F)_a = DG_{F(a)} \cdot DF_a.$$

Constant Functions. If $U \subset \mathbb{R}^n$ is an open connected set, then the differentiable mapping $F : U \to \mathbb{R}^m$ is constant if and only if $DF_x = 0$ for all $x \in U$.

Mean Value Theorem. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable and $U$ contains the line segment $[a, b]$, then $\exists c \in [a, b]$ such that

$$f(b) - f(a) = f'(c)(b - a) = \nabla f(c) \cdot (b - a).$$

(Does not generalize to $F : \mathbb{R}^n \to \mathbb{R}^m$.)

Equality of Mixed Partial. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is $C^2$, then $D_i D_j f = D_j D_i f$. 

Taylor Expansion. Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be \( C^2 \), then
\[
f(x_0 + h) = f(x_0) + Df_{x_0}(h) + \frac{1}{2} h^T \text{Hess}_{x_0}(f) h + R_2(x_0, h)
\]
where \( \lim_{h \to 0} \frac{R_2(x_0, h)}{|h|^2} = 0 \).

Second Derivative Test for Local Extrema. Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be \( C^3 \) and suppose that \( x_0 \in U \) is a critical point of \( f \).

1. If \( \text{Hess}_{x_0} \) is positive-definite, then \( x_0 \) is a relative minimum of \( f \).
2. If \( \text{Hess}_{x_0} \) is negative-definite, then \( x_0 \) is a relative maximum.

Lagrange Multipliers. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g_1, \ldots, g_m : \mathbb{R}^n \rightarrow \mathbb{R} \) be \( C^1 \) functions. If \( f \) has a local extremum at \( x_0 \in \{ x : g_1(x) = \ldots = g_m(x) = 0 \} \) and the vectors \( \nabla g_1(x_0), \ldots, \nabla g_m(x_0) \) are linearly independent, then there are real numbers \( \lambda_1, \ldots, \lambda_m \) such that
\[
\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \ldots + \lambda_m \nabla g_m(x_0)
\]

2. Vector Calculus

Basic Vector Operators. Let \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( F = (F_1, F_2, F_3) \) be a vector field and let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be a scalar field.

1. Gradient: \( \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \)
2. Divergence: \( \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \)
3. Curl: \( \nabla \times F(x) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \)
4. Gradients are curl-free: \( \nabla \times (\nabla f) = 0 \)
5. Curls are divergence-free: \( \nabla \cdot (\nabla \times F) = 0 \)

Let \( \gamma(t) = (x(t), y(t), z(t)) \) be a \( C^1 \) parametrized curve for \( a \leq t \leq b \).

Path Integral. If \( f : \gamma([a, b]) \rightarrow \mathbb{R} \) is continuous we define
\[
\int_\gamma f \, ds := \int_a^b f(\gamma(t)) \| \gamma'(t) \| \, dt.
\]

Line Integral. If \( F = (F_1, F_2, F_3) \) is a continuous vector field on \( \gamma([a, b]) \) we define
\[
\int_\gamma F \cdot ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_\gamma F_1 \, dx + F_2 \, dy + F_3 \, dz
\]

Fundamental Theorem of Calculus for Line Integrals.
\[
\int_\gamma \nabla f \cdot ds = f(\gamma(b)) - f(\gamma(a))
\]
Let \( \Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \Phi(u, v) = (x, y, z) \) be a \( C^1 \) parametrization of a surface. We define the tangent vectors
\[
T_u := \frac{\partial \Phi}{\partial u} \text{ and } T_v := \frac{\partial \Phi}{\partial v}
\]
and say that a surface is **regular** wherever \( N := T_u \times T_v \neq 0 \). As for path integrals, we can integrate a scalar \( f : \Phi(D) \rightarrow \mathbb{R} \) over the surface:
\[
\int_{\Phi} f \, dS = \iint_D f(\Phi(u, v)) \|T_u \times T_v\| \, du \, dv.
\]

**Surface Integral.** If \( F \) is a continuous vector field on \( \Phi(D) \), we define
\[
\int_{\Phi} F \cdot dS = \iint_D F(\Phi(u, v)) \cdot (T_u \times T_v) \, du \, dv.
\]

**Green’s Theorem.** Let \( D \subset \mathbb{R}^2 \) be an oriented 2-manifold-with-boundary and suppose \( \partial D \) is positively oriented. If \( P \, dx + Q \, dy \) be a \( C^1 \) differential 1-form in a neighbourhood of \( D \), then
\[
\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_{\partial D} P \, dx + Q \, dy.
\]

**Stoke’s Theorem.** Let \( D \) be an oriented compact 2-manifold-with-boundary in \( \mathbb{R}^3 \) and let \( N \) and \( T \) be the positively oriented unit normal and unit tangent vector fields on \( D \) and \( \partial D \) respectively. If \( F \) is a \( C^1 \) vector field on an open set containing \( D \), then
\[
\iint_D (\nabla \times F) \cdot N \, dA = \int_{\partial D} F \cdot T \, ds.
\]

**Gauss' Divergence Theorem.** Let \( F \) be a \( C^1 \) vector field defined on a neighbourhood of the compact oriented smooth \( n \)-manifold with boundary \( V \subset \mathbb{R}^n \). Then
\[
\iiint_V \nabla \cdot F = \iint_{\partial V} F \cdot N \, dA
\]
where \( N \) is the unit outer normal vector field on the positively oriented \( \partial V \).

**Conservative Vector Fields.** Let \( F \) be a \( C^1 \) vector field defined in a contractible region of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). We say \( F \) is conservative if any of the following equivalent conditions hold:

1. \( \int_\gamma F \cdot ds = 0 \) for every closed simple curve \( \gamma \)
2. \( \vec{F} = \nabla f \) for some function \( f \)
3. \( dF = \nabla \times F = 0 \)

3. **Integration in \( \mathbb{R}^n \rightarrow \mathbb{R}^m \)**

**Change of Variable.** Let \( T : D \rightarrow T(D) \) be \( C^1 \)-invertible on the interior of \( T(D) \), then
\[
\int_{T(D)} f(x, y) \, dx \, dy = \int_D f(u(x, y), v(x, y)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.
\]

**Cylindrical:** \( dx \, dy \, dz = r \, dr \, d\theta \, dz \). **Spherical:** \( dx \, dy \, dz = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi \).
**Fubini-Tonelli Theorem.** Let $f$ be $A \times B$ measurable. If any one of the following three conditions hold:

$$\int_{A \times B} |f(x,y)| \, d(x,y) < \infty, \int_{A} \left( \int_{B} |f(x,y)| \, dy \right) \, dx < \infty, \text{ or } \int_{B} \left( \int_{A} |f(x,y)| \, dx \right) \, dy < \infty,$$

then

$$\int_{A \times B} f(x,y) \, d(x,y) = \int_{A} \left( \int_{B} f(x,y) \, dy \right) \, dx = \int_{B} \left( \int_{A} f(x,y) \, dx \right) \, dy.$$

(Tonelli’s Theorem states that for positive functions the iterated integrals of $|f(x,y)|$ converge/diverge together.)

**Dominated Convergence Theorem.** Let $f_n$ be a sequence of measurable functions converging pointwise on $A$ to a function $f$. If $\exists g \geq 0$ such that $|f_n| \leq g$ and $\int_A g < \infty$ then $\int_A f_n \to \int_A f$.

Another criterion to guarantee the result of the Dominated Convergence Theorem is that the sequence $f_n$ be uniformly convergent.

**Differentiating Under the Integral Sign.** Let $f : A \times J \to \mathbb{R}$ be continuous and let $\frac{\partial}{\partial t} f(x,t)$ be uniformly continuous on $A \times J$, then

$$\frac{\partial}{\partial t} \int_A f(x,t) \, dx = \int_A \frac{\partial}{\partial t} f(x,t) \, dx.$$

**Improper Integration.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be locally integrable and let $U_k$ be an approximating sequence for $U$. We say that $f(x)$ is absolutely integrable if

$$\int_U |f| := \lim_{k \to \infty} \int_{U_k} |f| < \infty.$$

In this case, the following limit exists and is used as a definition of the improper integral

$$\int_U f := \lim_{k \to \infty} \int_{U_k} f.$$

**Comparison Test for Improper Integrals.** Suppose that $f$ and $g$ are locally integrable on $U$ with $0 \leq f \leq g$. If $g$ is absolutely integrable on $U$, then so is $f$.

Useful test case:

$$\int_1^\infty \frac{1}{x^p} \, dx = \begin{cases} \infty & \text{if } p \leq 1 \\ \frac{1}{1-p} & \text{if } p > 1 \end{cases} \quad \& \quad \int_0^1 \frac{1}{x^p} \, dx = \begin{cases} \infty & \text{if } p \geq 1 \\ \frac{1}{1-p} & \text{if } p < 1 \end{cases}$$

For functions of a single variable, we can sometimes assign a value even if the function is not absolutely integrable. If $f$ is continuous on $[a,x_0] \cup [x_0,b]$, we define the **Cauchy Principal Value**:

$$PV \int_a^b f(x) \, dx := \lim_{\epsilon \to 0} \left( \int_a^{x_0-\epsilon} f(x) \, dx + \int_{x_0+\epsilon}^b f(x) \, dx \right).$$
4. Fundamental Real Analysis

**Ratio Test (for sequences).** If \((x_n)\) is a sequence of positive real numbers with \(\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = L < 1\) then \(x_n \to 0\).

**Monotone Convergence Theorem.** A monotone sequence of real numbers is convergent \(\iff\) it is bounded. Moreover, if it is decreasing (increasing) it converges to its infimum (supremum).

**Bolzano-Weierstrass Theorem.** A bounded sequence of real numbers has a convergent subsequence.

**Cauchy Convergence Criterion.** A sequence of real numbers is convergent if and only if it is Cauchy.

**Contractive Sequence Criterion.** A sequence \((x_n)\) of real numbers is contractive if
\[
|x_{n+2} - x_{n+1}| \leq K|x_{n+1} - x_n|
\]
for some \(0 < K < 1\). Such a sequence is Cauchy and therefore convergent.

**Cauchy-Schwarz Inequality.** For \(x, y\) in an inner-product space:
\[
\left\| \langle x, y \rangle \right\|^2 \leq \langle x, x \rangle \langle y, y \rangle = \left\| x \right\|^2 \left\| y \right\|^2.
\]

**Minkowski Inequality.** Let \(1 \leq p\) and \(a_k, b_k \in \mathbb{C}\), then
\[
\left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}}.
\]

**Holder Inequality.** If \(\frac{1}{p} + \frac{1}{q} = 1\) and \(a_k, b_k \in \mathbb{C}\), then
\[
\sum_{k=1}^{n} |a_k \cdot b_k|^p \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}}.
\]

**Absolute Convergence Tests.** Let \(a_n\) and \(b_n\) be sequences of positive real numbers.

1. **Limit Comparison Test:**
\[
\sum a_n < \infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{b_n}{a_n} < \infty \implies \sum b_n < \infty
\]
\[
\sum a_n = \infty \quad \text{and} \quad \liminf_{n \to \infty} \frac{b_n}{a_n} > 0 \implies \sum b_n < \infty.
\]

2. **Ratio Test:**
\[
\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \implies \sum a_n < \infty \quad \& \quad \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 1 \implies \sum a_n = \infty
\]

3. **Root Test:**
\[
\limsup_{n \to \infty} (a_n)^{1/n} < 1 \implies \sum a_n < \infty \quad \& \quad \limsup_{n \to \infty} (a_n)^{1/n} > 1 \implies \sum a_n = \infty
\]
Alternating Series Test. If \( a_n \geq 0, a_{n+1} \leq a_n \) and \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

Integral Test. If \( f : [c, \infty[ \to [0, \infty[ \) is a positive decreasing continuous function with \( f(n) = a_n \) for all \( n \geq c \), then
\[
\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_c^{\infty} f(x) \, dx < \infty.
\]

Continuous Functions. The function \( f : X \to Y \) between metric spaces \( X \) and \( Y \) is continuous at \( x_0 \in X \) if one of the following conditions hold:
\begin{enumerate}
  \item \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \)
  \item \( f(x_0) \in U \) open \( \implies f^{-1}(U) \) open (for all such \( U \))
  \item \( x_n \to x \implies f(x_n) \to f(x) \) (for all such sequences \( x_n \))
\end{enumerate}

Continuous Functions on Compact Sets. Let \( X \) be a compact metric space.
\begin{enumerate}
  \item If \( f : X \to \mathbb{R} \) is continuous, \( f(X) \) is a bounded set and \( f \) attains its supremum and infimum.
  \item If \( Y \) is any metric space and \( f : X \to Y \) is continuous, then \( f \) is uniformly continuous.
\end{enumerate}

Taylor Expansion. Let \( f : [a, b] \to \mathbb{R} \) be \( n \)–times differentiable. If \( x, x_0 \in [a, b] \), then
\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)
\]
where \( \frac{R_n(x)}{(x-x_0)^n} \to 0 \) as \( x \to x_0 \). There are other forms of the remainder:
\begin{enumerate}
  \item If \( f^{(n+1)} \) exists in \( ]a, b[ \), we may write \( R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \) for some \( \xi \in ]x_0, x[ \).
  \item If \( f^{(n+1)} \) is integrable on \( [a, b] \), we may write \( R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt \).
\end{enumerate}

Mean Value Theorem. Let \( f : [a, b] \to \mathbb{R} \) be continuous. If \( f \) is differentiable in \( ]a, b[ \), then there is some \( \xi \in ]a, b[ \) such that
\[
f(b) - f(a) = f'(\xi)(b-a).
\]

Integral Mean Value Theorem. If \( f, g : [a, b] \to \mathbb{R} \) where \( f(x) \) is continuous and \( 0 \leq g(x) \) is integrable, then there is some \( \xi \in [a, b] \) such that
\[
\int_{a}^{b} f(x)g(x) \, dx = f(\xi) \int_{a}^{b} g(x) \, dx.
\]

Intermediate Value Theorem. Let \( f : [a, b] \to \mathbb{R} \) be continuous. If \( f(a) < k < f(b) \), then there is some \( \xi \in [a, b] \) such that \( f(\xi) = k \).

Darboux’s Theorem. Let \( f : [a, b] \to \mathbb{R} \) be differentiable. If \( f'(a) < k < f'(b) \) then there is some \( \xi \in ]a, b[ \) such that \( f'(\xi) = k \).
The Class of Riemann Integrable Functions. Let \( f, g : [a, b] \to \mathbb{R} \) be bounded functions (an integrable function is necessarily bounded).

1. (Lebesgue) \( f(x) \) is integrable \( \iff \) it is continuous almost everywhere.
2. (Composition) \( f(x) \) integrable \& \( \varphi(y) \) continuous \( \implies \varphi \circ f \) integrable
3. (Absolute Value) \( f(x) \) integrable \( \iff |f(x)| \) integrable
4. (Product) \( f(x) \) \& \( g(x) \) integrable \( \implies f(x) \cdot g(x) \) integrable

Uniformly Continuous Functions. Let \( f : A \to \mathbb{R} \) be a function.

1. \( f \) is Lipschitz \( \implies f \) uniformly continuous
2. \( f \) uniformly continuous \( \iff (f(x_n)) \) is Cauchy whenever \( (x_n) \) is Cauchy in \( A \)

Continuous Extension Theorem. A function \( f : [a, b] \to \mathbb{R} \) is uniformly continuous \( \iff \) it admits a continuous extension \( f : [a, b] \to \mathbb{R} \).

Weierstrass Approximation Theorem. Let \( f : [a, b] \to \mathbb{R} \) be continuous. Given \( \varepsilon > 0 \), there is a polynomial \( p_\varepsilon(x) \) such that \( |f(x) - p_\varepsilon(x)| < \varepsilon \) for all \( x \in [a, b] \).

Banach Fixed Point Theorem. Let \( X \) be a complete metric space. If \( T : X \to X \) satisfies \( |T(x) - T(y)| \leq K|x - y| \) for some \( K < 1 \) (i.e. it is a contraction), then \( T \) has a unique fixed point.

Newton’s Method. Let \( f : [a, b] \to \mathbb{R} \) be twice differentiable and suppose that \( f(a)f(b) < 0 \). If \( |f'(x)| \geq m > 0 \) and \( |f''(x)| \leq M \) for all \( x \in [a, b] \), there is a subinterval \( I \subset [a, b] \) containing a root of \( f(r) = 0 \) such that for any \( x_1 \in I \) the sequence

\[
x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}
\]

converges to \( r \).

Inverse Function Theorem. Let \( f : [a, b] \to \mathbb{R} \) be strictly monotone and continuous. If \( f \) is differentiable at \( c \) and \( f'(c) \neq 0 \), then \( f^{-1} \) is differentiable at \( f(c) \) and \( (f^{-1})'(f(c)) = \frac{1}{f'(c)} \).

More generally, let \( F : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be \( C^1 \) and let \( a \in U \). If \( DF(a) \) is invertible, then \( F \) is invertible in a neighbourhood of \( F(a) \).

Given \( G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) and \( F : U \subset \mathbb{R}^m \to \mathbb{R}^n \), we say that \( y = F(x) \) solves the equation \( G(x, y) = 0 \) in a neighbourhood \( W \) if \( G(x, y) = 0 \iff y = F(x) \) whenever \( (x, y) \in W \).

Implicit Function Theorem. Let \( G : U \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) be \( C^1 \) in a neighbourhood of \( (a, b) \) with \( G(a, b) = 0 \). If \( \frac{\partial G}{\partial y}(a, b) \) is invertible, then there is a function \( F : U \subset \mathbb{R}^m \to \mathbb{R}^n \) which solves \( G(x, y) = 0 \) in some neighbourhood of \( (a, b) \). Moreover, since

\[
0 = \frac{d}{dx} G(x, F(x)) = \frac{\partial G}{\partial x} \frac{dx}{dx} + \frac{\partial G}{\partial y} \frac{d}{dx} F(x) = \frac{\partial G}{\partial x}(x, y) + \frac{\partial G}{\partial y}(x, y) F'(x)
\]

we can implicitly compute \( F'(x) = -\left[ \frac{\partial G}{\partial y}(x, y) \right]^{-1} \frac{\partial G}{\partial x}(x, y) \).
5. Analysis on Functions

**Weierstrass M-test.** Let \( M_k \geq 0 \) be a sequence of complex numbers for which \( \sum M_k < \infty \). If \( f_n : X \subset \mathbb{C} \rightarrow \mathbb{C} \) is a sequence of functions such that \( |f_n(x)| \leq M_n \) for all \( x \in X \) and \( n \in \mathbb{N} \), then \( \sum f_n(x) \) converges absolutely and uniformly on \( X \). In this case we can integrate and differentiate the series of functions term by term.

**Differentiability of Limit Functions.** If \( f_n(x) \) is a sequence of differentiable real-valued functions defined on an open interval, then

\[
\left[ f_n \xrightarrow{\text{pointwise}} f \quad \& \quad f'_n \xrightarrow{\text{uniformly}} g \right] \implies \left[ f_n \xrightarrow{\text{uniformly}} f \quad \& \quad f' = g \right].
\]

Counterexample: \( f_n(x) = \frac{1}{n} \sin(nx) \).

**Power Series.** Every power series \( \sum_k a_k z^k \) has a radius of convergence \( 0 \leq \rho \leq +\infty \) for which \( \sum_k a_k z^k \) converges absolutely if \( |z| < \rho \) and diverges if \( |z| > \rho \). Moreover, the series converges uniformly on compact subsets.

1. The radius of convergence is given by the **Cauchy-Hadamard Formula**

\[
\rho = \frac{1}{\limsup |a_k|^{1/k}}.
\]

2. Power series can be added and multiplied. In this case the radius of convergence may shrink to the minimum of the two radii.

3. If \( a_0 \neq 0 \) we may invert a power series (or compose it with another) formally. In this case, we just know that the radius of convergence remains strictly positive.

4. Uniform convergence on compact subdisks allows power series to be differentiated and integrated term by term without affecting the radius of convergence. This yields a formula for its terms:

\[
a_k = \frac{f^{(k)}(0)}{k!}
\]

which shows that a power series is uniquely determined by the function it represents.

**Important Power Series.**

1. \( \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \), \( |z| < 1 \).

2. \( \log(1-z) = \sum_{k=0}^{\infty} \frac{z^k}{k+1} \), \( |z| < 1 \).

3. \( e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \), \( |z| < \infty \).

4. \( \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \), \( |z| < \infty \).

5. \( \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \), \( |z| < \infty \).

**Dini’s Theorem.** If \( f_n : A \rightarrow \mathbb{R} \) is a sequence of continuous functions defined on a compact metric space, then

\[
f_n \xrightarrow{\text{pointwise}} f \quad \& \quad f_n(x) \geq f_{n+1}(x) \implies f_n \xrightarrow{\text{uniformly}} f.
\]

**Arzela-Ascoli Theorem.** Let \( f_n : A \rightarrow \mathbb{C} \) is a sequence of continuous functions defined on a compact metric space. If \( (f_n) \) is pointwise bounded \( \& \) equiuniformly, then it is uniformly bounded and contains a uniformly convergent subsequence.
Fourier Series. Let $f(x)$ be defined in $]-L,L[$ and determined outside this interval by $f(x+2L) = f(x)$. The Fourier Series Expansion of $f(x)$ is by definition

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = \sum_{n=1}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}}$$

where for any $\xi \in \mathbb{R}$ we have

$$a_n := \frac{1}{L} \int_{\xi}^{\xi+2L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad b_n := \frac{1}{L} \int_{\xi}^{\xi+2L} f(x) \sin \frac{n\pi x}{L} \, dx$$

and

$$c_n := \frac{1}{2L} \int_{\xi}^{\xi+2L} f(x)e^{-i\frac{n\pi x}{L}} \, dx.$$

Dirichlet Conditions. Let $f(x)$ be defined and single-valued except possibly at a finite number of points in $]-L,L[$ and suppose that $f(x+2L) = f(x)$.

1. If $f(x)$ and $f'(x)$ are both piecewise continuous, then the Fourier series converges pointwise to $\frac{f(x+)+f(x-)}{2}$.
2. If $f$ is continuous and $f'$ is piecewise continuous, then the Fourier series converges uniformly to $f$.

Bessel Inequality and Parseval Identity. If $\{\varphi_k\}_{k=0}^{\infty}$ be an orthonormal system in an inner-product space $V$, then:

1. $\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle^2$ converges and $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2 \leq \langle f, f \rangle = ||f||^2$ for all $f \in V$
2. $\{\varphi_k\}_{k=0}^{\infty}$ is complete $\iff$ $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2 = \langle f, f \rangle$ for all $f \in V$.

6. Basic Properties of Analytic Functions

For $z \neq 0$ we have the following multivalued functions:

1. $\log z := \log ||z|| + i \arg z = \log ||z|| + i \arg z + 2\pi mi$, $m \in \mathbb{Z}$ and $-\pi < \arg z \leq \pi$
2. $z^n := e^{\log z} = |z|^n e^{i\alpha \arg z}$

along with their principal branches (other branches specified by choice of $m \in \mathbb{Z}$)

1. Log $z := \log ||z|| + i \arg z$ with branch cut $z \notin (-\infty, 0]$.
2. $z^n = |z|^n e^{i\alpha \arg z}$ with branch cut $z \notin [0, \infty]$.

Warning $\log z^n \neq n \log z$ while $\log z^{1/n} = \frac{1}{n} \log z$ as multivalued sets

Phase Change Lemma. Let $g(z)$ be a single valued function, continuous in a neighbourhood of $z_0$. For any branch of $(z-z_0)^{\alpha}$, the function $f(z) = (z-z_0)^{\alpha}g(z)$ is multiplied by the phase factor $e^{2\pi i \alpha}$ when $z$ traverses a complete circle about $z_0$ in the positive direction.

Cauchy-Riemann Equations. The function $f = u + iv$ defined on the domain $\mathcal{D}$ is analytic if and only if the following two conditions hold:

1. The functions $u(x,y)$ and $v(x,y)$ have continuous first-order partial derivatives.
Detecting Poles.

Riemann’s Theorem on Removable Singularities. Let \( f_k(z) \) be analytic on \( \mathcal{D} \).

1. If \( (f_k(z)) \) uniformly converges to \( f(z) \), then \( f(z) \) is analytic on \( \mathcal{D} \).
2. If \( (f_k(z)) \) normally converges to \( f(m)(z) \), then \( f_k^{(m)}(z) \) normally converges to \( f^{(m)}(z) \).

Here a sequence \( (f_k(z)) \) of analytic functions on \( \mathcal{D} \) converges normally to the analytic function \( f(z) \) if it converges uniformly to \( f(z) \) on every closed disk contained in \( \mathcal{D} \).

Proof. Morera’s Theorem.

Laurent Decomposition. Let \( f(z) \) be analytic on the annulus \( 0 < \rho < |z - z_0| < \sigma \leq \infty \).

1. The function can be decomposed as a sum \( f(z) = f_\sigma(z) + f_\rho(z) \) where \( f_\sigma(z) \) is analytic for \( |z - z_0| < \sigma \) and \( f_\rho(z) \) is analytic for \( |z - z_0| > \rho \) (including at \( \infty \)).
2. If we normalize the decomposition so that \( f_\rho(\infty) = 0 \), then the decomposition is unique.

Expressing \( f_\sigma \) and \( f_\rho \) in terms of power series, we obtain the Laurent series

\[
f(z) = \sum_{k=-\infty}^{-1} a_k(z - z_0)^k + \sum_{k=0}^{\infty} a_k(z - z_0)^k = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k, \quad \rho < |z - z_0| < \sigma
\]

converging absolutely (and uniformly on compact sub-annuli) with coefficients

\[
a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}} \, dz.
\]

Proof. Uniqueness = Liouville’s Theorem, existence = the Cauchy Integral Formula.

Classification of Isolated Singularities. Let \( z_0 \) be an isolated singularity of

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k, \quad 0 < |z - z_0| < \sigma.
\]

1. The singularity is removable if \( a_k = 0 \) for all \( k < 0 \).
2. The singularity is a pole of order \( N \) if \( a_{-N} \neq 0 \) and \( a_k = 0 \) for all \( k < -N \).
3. Otherwise, the singularity is essential.

If \( z_0 = \infty \) we apply the above definitions to \( f(1/z) \).

Riemann’s Theorem on Removable Singularities.

isolated singularity \( z_0 \) of \( f(z) \) removable \( \iff \) \( f(z) \) bounded in neighbourhood of \( z_0 \)

Detecting Poles. Let \( z_0 \) be an isolated singularity of \( f(z) \).

1. \( z_0 \) is a pole \( \iff \) \( |f(z)| \to \infty \) as \( z \to \infty \)
2. \( z_0 \) is a pole of order \( N \) \( \iff \) \( f(z) = g(z)/(z-z_0)^N \) where \( g(z) \) analytic at \( z_0 \) and \( g(z_0) \neq 0 \)
Casorati-Weierstrass Theorem. If $z_0$ is an essential singularity of $f(z)$, then for every $w_0 \in \mathbb{C}$ there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.

Picard’s Theorem. There are two:

1. **Little**: The image of an entire & non-constant function misses at most one point of $\mathbb{C}$.

2. **Big**: If an entire function has an essential singularity at $z_0$, then it assumes all possible complex values with at most a single exception infinitely often in any neighbourhood of $z_0$.

Partial Fractions Decomposition. A meromorphic function on $\mathbb{C}^*$ is rational. Every rational function has a partial fraction decomposition expressing it as the sum of a polynomial in $z$ and its principal parts at each of its poles in the finite complex plane.

Uniqueness Principle. Let $f(z)$ and $g(z)$ be analytic on $\mathcal{D}$.

1. If $f(z)$ is not identically zero on $\mathcal{D}$, then the zeroes of $f(z)$ are isolated.

2. If $f(z) = g(z)$ for all $z$ belonging to a subset $D$ of $\mathcal{D}$ containing a non-isolated point, then $f(z) = g(z)$ on $\mathcal{D}$.

Analytic Continuation. Let $f(z) = \sum a_k(z - z_0)^k$ for $|z - z_0| < \rho$. We say $f(z)$ has an analytic continuation along a path $\gamma(t)$ parametrized by $a \leq t \leq b$ if for every $t$ there is a convergent power series

$$f_t(z) := \sum_{k=0}^{\infty} a_k(t)(z - \gamma(t))^n, \text{ for } |z - \gamma(t)| < \rho(t)$$

with $f_a(z) = f(z)$ and $f_b(z) = f_t(z)$ wherever their disks of convergence intersect.

Such an analytic continuation is unique. Further, the coefficients $a_k(t)$ and radii of convergence $\rho(t)$ depend continuously on $t$. The Monodromy Theorem ensures that analytic continuations along homotopic paths upon which $f(z)$ admits analytic continuations must coincide.

7. Harmonic Functions and Bounds on Analytic Functions

A function $u(x, y)$ is harmonic if $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the partial derivatives are continuous. If $f = u + iv$ is analytic, then $u(x, y)$ and $v(x, y)$ are harmonic as a consequence of the Cauchy-Riemann equations.

Harmonic Conjugate. Let $\mathcal{D}$ be a simply connected domain. If $u(x, y)$ is a real valued harmonic function on $\mathcal{D}$, then there is a unique (up to adding a constant) harmonic conjugate $v(x, y)$ such that $f = u + iv$ is analytic on $\mathcal{D}$.

Mean Value Property. Let $u(z)$ be harmonic on $\mathcal{D}$. If $\{|z - z_0| < \rho\} \subset \mathcal{D}$, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta \text{ for every } 0 < r < \rho.$$
It turns out that a continuous function on \( \mathcal{D} \) has the mean value property if and only if it is harmonic.

**Strict Maximum Principle.** Let \( u(z) \) be a real valued harmonic function on \( \mathcal{D} \) with \( u(z) \leq M \) for all \( z \in \mathcal{D} \). If \( u(z_0) = M \) for some \( z_0 \in \mathcal{D} \), then \( u(z) \equiv M \).

Proof. Mean value property \( \implies \{ z : u(z) = M \} \) open. Continuity \( \implies \{ z : u(z) < M \} \) is open. Contradicts connectedness of \( \mathcal{D} \). \qed

**Maximum Modulus Principle.** Let \( h(z) \) be a complex-valued harmonic (analytic) function on \( \mathcal{D} \).

1. If \( |h(z)| \leq M \) for all \( z \in \mathcal{D} \) and \( |h(z_0)| = M \) for some \( z_0 \in \mathcal{D} \), then \( h(z) \) is constant on \( \mathcal{D} \).
2. Suppose further that \( \mathcal{D} \) is bounded and \( h(z) \) extends continuously to \( \partial \mathcal{D} \). If \( |h(z)| \leq M \) for all \( z \in \partial \mathcal{D} \), then \( |h(z)| \leq M \) for all \( z \in \mathcal{D} \).

**Cauchy Estimates.** Let \( f(z) \) be analytic for \( |z - z_0| \leq \rho \). If \( |f(z)| \leq M \) \( |z - z_0| = \rho \), then for every \( m \geq 0 \) we have the following bound:

\[
|f^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M.
\]

**Liouville’s Theorem.** A bounded entire function is constant. Here, an entire function is one who is analytic on the entire complex plane.

8. **Complex Integration**

**Fundamental Theorem of Calculus.** As usual, there are two parts:

1. Let \( f(z) \) be continuous on \( \mathcal{D} \). If \( F(z) \) is a primitive of \( f(z) \), then

\[
\int_A^B f(z) \, dz = F(B) - F(A)
\]

and the integral can be taken over any path in \( \mathcal{D} \) from \( A \) to \( B \).

2. Let \( \mathcal{D} \) be simply connected. If \( f(z) \) is analytic on \( \mathcal{D} \), then a primitive is given by

\[
F(z) := \int_{z_0}^z f(w) \, dw, \text{ for } z \in \mathcal{D}
\]

where \( z_0 \) is any fixed point and the integral can be taken along any path in \( \mathcal{D} \).

**Cauchy’s Theorem.** Let \( \mathcal{D} \) be a bounded domain with piecewise smooth boundary. If \( f(z) \) is analytic on \( \mathcal{D} \) and extends smoothly to \( \partial \mathcal{D} \), then \( \int_{\partial \mathcal{D}} f(z) \, dz = 0 \).

Proof. A \( C^1 \) \( f(z) \) on \( \mathcal{D} \) is analytic if and only if the differential \( f(z) \, dz \) is closed + Green’s Theorem. \qed
Cauchy Integral Formula. Let $\mathcal{D}$ be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on $\mathcal{D}$ which extends smoothly to $\partial \mathcal{D}$, then for every $z_0 \in \mathcal{D}$ and $m \geq 0$ we have:

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{\partial \mathcal{D}} \frac{f(z)}{(z-z_0)^{m+1}} dz$$

Morera’s Theorem. Let $f(z)$ be a continuous function on $\mathcal{D}$. If $\int_{\partial T} f(z) \, dz = 0$ for every closed triangle $T$ contained in $\mathcal{D}$, then $f(z)$ is analytic on $\mathcal{D}$.

Residue Theorem. Let $\mathcal{D}$ be a bounded domain with piecewise smooth $\partial \mathcal{D}$. If $f(z)$ is analytic on $\mathcal{D} \cup \partial \mathcal{D}$ except at the isolated singularities $z_1, \ldots, z_m \in \mathcal{D}$, then

$$\int_{\partial \mathcal{D}} f(z) \, dz = 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j].$$

If $\mathcal{D}$ is an exterior domain with piecewise smooth $\partial \mathcal{D}$ and we let $a_{-1}$ be the coefficient of $\frac{1}{z}$ in the Laurent expansion of $f(z)$ convergent for $|z| > R$, then

$$\int_{\partial \mathcal{D}} f(z) \, dz = -2\pi ia_{-1} + 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j].$$

Equivalently, $-a_{-1} = \text{Res}[f(z), \infty] = \text{Res}[-\frac{1}{z} f(\frac{1}{z}), 0]$.

Computing Residues. Two tricks:

1. If $f(z)$ has a pole of order $n$ at $z_0$, then

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)].$$

2. If $f(z)$ and $g(z)$ are analytic at $z_0$ and $g(z)$ has a simple zero at $z_0$, then

$$\text{Res} \left[ \frac{f(z)}{g(z)}, z_0 \right] = \frac{f(z_0)}{g'(z_0)}.$$

Jordan’s Lemma. If $\Gamma_R$ is the contour $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$ and $a > 0$, then

$$\int_{\Gamma_R} |e^{iaz}| \, |dz| < \frac{\pi}{a}.$$

Fractional Residue Theorem. Let $z_0$ be a simple pole of $f(z)$ and let $C_\epsilon$ be an arc of the circle $|z-z_0| = \epsilon$ of angle $\alpha$, then

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) \, dz = \alpha i \cdot \text{Res}[f(z), z_0].$$

Argument Principle. Let $\mathcal{D}$ be bounded with piecewise smooth $\partial \mathcal{D}$. If $f(z)$ is meromorphic on $\mathcal{D}$ and extends to be analytic on $\partial \mathcal{D}$ with $f(z) \neq 0$ for all $z \in \partial \mathcal{D}$, then

$$\int_{\partial \mathcal{D}} d \log f(z) = \int_{\partial \mathcal{D}} \frac{f'(z)}{f(z)} \, dz = 2\pi i \cdot [N_0 - N_\infty].$$
Splitting up $\log f(z) = \log |f(z)| + i \arg f(z)$ we obtain the following restatement

$$\int_{\partial \mathcal{D}} d\arg f(z) = 2\pi \cdot [N_0 - N_\infty].$$

**Rouché’s Theorem.** Let $\mathcal{D}$ be bounded with piecewise smooth $\partial \mathcal{D}$. If $f(z)$ and $h(z)$ are analytic on $\mathcal{D} \cup \partial \mathcal{D}$ with $|h(z)| < |f(z)|$ on $\partial \mathcal{D}$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeroes in $\mathcal{D}$ (with multiplicities).

**Open Mapping Theorem.** A non-constant analytic map on a domain $\mathcal{D}$ is open.

**Fourier Transform.** If $f$ be absolutely integrable on $\mathbb{R}$, its Fourier Transform is well defined by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixw} dx.$$

If $\hat{f}$ is absolutely integrable and $f$ is continuous at $x$, then the Fourier Inversion Formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{ixw} dw.$$

**Riemann-Lebesgue Lemma.**

$$f \in L^1 \implies \lim_{n \to \infty} \int f(x)e^{inx} dx = 0.$$

9. **Complex Mappings**

**Inverse Function Theorem.** If $f(z)$ is analytic on $\mathcal{D}$ and $f'(z_0) \neq 0$, then $f$ has an analytic inverse $f^{-1}$ in some neighbourhood of $f(z_0)$ where $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$.

**Conformal Mapping Theorem.**

$$[f(z) \text{ analytic at } z_0 \& f'(z_0) \neq 0] \implies f(z) \text{ conformal at } z_0$$

**Schwarz Lemma.** Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$, then:

1. $|f(z)| \leq |z|$ for $|z| < 1$ where equality at any $z_0 \neq 0$ implies that $f(z) = e^{i\theta}z$
2. $|f'(0)| \leq 1$ where equality implies that $f(z) = e^{i\theta}z$.

**Mobius Transformations.** A Mobius Transformation is a function of the form

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0$$

1. Given distinct $z_0, z_1, z_2 \in \mathbb{C}^*$ there is a Mobius transformation mapping them onto any distinct $w_0, w_1, w_2 \in \mathbb{C}^*$.
2. The group of Mobius transformation is generated by dilations, translations and inversions.
3. A Mobius transformation maps circles to circles in $\mathbb{C}^*$. 
Important Conformal Maps.

\( \mathbb{H} = \{ \Im(z) > 0 \} \), \( \mathbb{D} = \{ |z| < 1 \} \) and \( S_\alpha = \{ 0 < \arg z < \alpha \} \)

1. \( \text{Aut} \mathbb{D} = \{ e^{i\theta} \frac{z-a}{1-\overline{a}z} : |a| < 1 \} \)
2. \( z \mapsto \frac{z-i}{z+i} \) maps \( \mathbb{H} \) conformally onto \( \mathbb{D} \)
3. \( w \mapsto i : \frac{1+w}{1-w} \) maps \( \mathbb{D} \) conformally onto \( \mathbb{H} \)
4. \( z \mapsto z^{\frac{\alpha}{\pi}} \) maps \( S_\alpha \) onto \( \mathbb{H} \)
5. \( z \mapsto \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)} \) sends \( z_0 \mapsto 0 \), \( z_1 \mapsto 1 \) and \( z_2 \mapsto \infty \)