## MATH 135: Randomized Exam Practice Problems

These are the warm-up exercises and recommended problems taken from all the extra practice sets presented in random order. The challenge problems have not been included.

- 1. Given the public RSA encryption key (e, n) = (5, 35), find the corresponding decryption key (d, n).
- 2. Let a, b, c be integers. Prove that if  $a \mid b$  then  $ac \mid bc$ .
- 3. Find a real cubic polynomial whose roots are 1 and i.
- 4. Let n, a and b be positive integers. Negate the following implication without using the word "not" or the  $\neg$  symbol (but symbols such as  $\neq$ ,  $\nmid$ , etc. are fine). *Implication:* If  $a^3 \mid b^3$ , then  $a \mid b$ .
- 5. Prove the following statements.
  - (a) There is no smallest positive real number.
  - (b) For every even integer n, n cannot be expressed as the sum of three odd integers.
  - (c) Let  $a, b \in \mathbb{Z}$ . If a is an even integer and b is an odd integer, then  $4 \nmid (a^2 + 2b^2)$ .
  - (d) For every integer m with  $2 \mid m$  and  $4 \nmid m$ , there are no integers x and y that satisfy  $x^2 + 3y^2 = m$ .
  - (e) The sum of a rational number and an irrational number is irrational.
  - (f) Let x be a non-zero real number. If  $x + \frac{1}{x} < 2$ , then x < 0.
- 6. Consider the following statement.

For all  $x \in \mathbb{R}$ , if  $x^6 + 3x^4 - 3x < 0$ , then 0 < x < 1.

- (a) Rewrite the given statement in symbolic form.
- (b) State the hypothesis of the implication within the given statement.
- (c) State the conclusion of the implication within the given statement.
- (d) State the converse of the implication within the given statement.
- (e) State the contrapositive of the implication within the given statement.
- (f) State the negation of the given statement without using the word "not" or the  $\neg$  symbol (but symbols such as  $\neq$ ,  $\ddagger$ , etc. are fine).
- (g) Prove or disprove the given statement.
- 7. For what values of c does 8x + 5y = c have exactly one solution where both x and y are strictly positive?
- 8. Let  $n \ge 2$  be an integer. Prove that

$$\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right)$$

- 9. Let  $a, b, c \in \mathbb{Z}$ . Prove that if gcd(a, b) = 1 and  $c \mid a$ , then gcd(b, c) = 1.
- 10. Suppose S and T are two sets. Prove that if  $S \cap T = S$ , then  $S \subseteq T$ . Is the converse true?
- 11. Let  $a, b, c \in \mathbb{Z}$ . Is the following statement true? Prove that your answer is correct.
  - $a \mid b$  if and only if  $ac \mid bc$ .

- 12. Find the smallest positive integer a such that  $5n^{13} + 13n^5 + a(9n) \equiv 0 \pmod{65}$  for all integers n.
- 13. Prove or disprove each of the following statements.
  - (a)  $\forall n \in \mathbb{Z}, \frac{(5n-6)}{3}$  is an integer.
  - (b) For every prime number p, p+7 is composite.
  - (c) There exists a natural number m < 123456 such that 123456m is a perfect square.
  - (d)  $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8).$
- 14. In each of the following cases, find all values of  $[x] \in \mathbb{Z}_m$ ,  $0 \le x < m$ , that satisfy the equation.
  - (a)  $[4][3] + [5] = [x] \in \mathbb{Z}_{10}$
  - (b)  $[7]^{-1} [2] = [x] \in \mathbb{Z}_{10}$
  - (c)  $[2][x] = [4] \in \mathbb{Z}_8$
  - (d)  $[3][x] = [9] \in \mathbb{Z}_{11}$
- 15. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.
  - (a) A sequence  $\{x_n\}$  is defined by  $x_1 = 3$ ,  $x_2 = 20$  and  $x_i = 5x_{i-1}$  for  $i \ge 3$ . Then, for all  $n \in \mathbb{N}$ ,  $x_n = 3 \times 5^{n-1}$ .

Let P(n) be the statement:  $x_n = 3 \times 5^{n-1}$ .

When n = 1 we have  $3 \times 5^0 = 3 = x_1$  so P(1) is true. Assume that P(k) is true for some integer  $k \ge 1$ . That is,  $x_k = 3 \times 5^{k-1}$  for some integer  $k \ge 1$ . We must show that P(k+1) is true, that is,  $x_{k+1} = 3 \times 5^k$ . Now

$$x_{k+1} = 5x_k = 5(3 \times 5^{k-1}) = 3 \times 5^k$$

as required. Since the result is true for n = k + 1, and so holds for all n by the Principle of Mathematical Induction.

(b) For all  $n \in \mathbb{N}$ ,  $1^{n-1} = 2^{n-1}$ .

Let P(n) be the statement:  $1^{n-1} = 2^{n-1}$ . When n = 1 we have  $1^0 = 1 = 2^0$  so P(1) is true. Assume that P(i) is true for all integers  $1 \le i \le k$  where  $k \ge 1$  is an integer. That is,  $1^{i-1} = 2^{i-1}$  for all  $1 \le i \le k$ . We must show that P(k+1) is true, that is,  $1^{(k+1)-1} = 2^{(k+1)-1}$  or  $1^k = 2^k$ . By our inductive hypothesis, P(2) is true so  $1^1 = 2^1$ . Also by our inductive hypothesis, P(k) is true so  $1^{k-1} = 2^{k-1}$ . Multiplying these two equations together gives  $1^k = 2^k$ . Since the result is true for n = k + 1, and so holds for all n by the Principle of Strong Induction.

16. State whether the given statement is true or false and prove or disprove accordingly.

- (a) For all  $a, b, c, x \in \mathbb{Z}$  such that c, x > 0, if  $a \equiv b \pmod{c}$  then  $a + x \equiv b + x \pmod{c + x}$ .
- (b) For all  $m \in \mathbb{N}$  and for all  $[a] \in \mathbb{Z}_m$  there exists a  $[b] \in \mathbb{Z}_m$  such that  $[b]^2 = [a]$ .
- 17. Prove that  $(\neg A) \lor B$  is logically equivalent to  $\neg (A \land \neg B)$ .

18. Suppose r is some (unknown) real number, where  $r \neq -1$  and  $r \neq -2$ . Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r(2^r)}{(r+1)(r+2)}$$

- 19. The Fibonacci sequence is defined as the sequence  $\{f_n\}$  where  $f_1 = 1$ ,  $f_2 = 1$  and  $f_i = f_{i-1} + f_{i-2}$  for  $i \ge 3$ . Use induction to prove the following statements.
  - (a) For  $n \ge 2$ ,

(b) Let 
$$a = \frac{1+\sqrt{5}}{2}$$
 and  $b = \frac{1-\sqrt{5}}{2}$ . For all  $n \in \mathbb{N}$ ,  $f_n = \frac{a^n - b^n}{\sqrt{5}}$ 

20. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \lor (B \wedge C) \equiv \neg((A \lor B) \implies \neg C)$$

 $f_1 + f_2 + \dots + f_{n-1} = f_{n+1} - 1$ 

- 21. Prove or disprove the following statements. Let a, b, c be fixed integers.
  - (a) If there exists an integer solution to  $ax^2 + by^2 = c$ , then  $gcd(a, b) \mid c$ .
  - (b) If  $gcd(a, b) \mid c$ , then there exists an integer solution to  $ax^2 + by^2 = c$ .
- 22. Write  $(\sqrt{3}+i)^4$  in standard form.
- 23. Prove that for every integer k,  $gcd(a, b) \leq gcd(ak, b)$ .
- 24. Prove that the product of any four consecutive integers is one less than a perfect square.
- 25. Let  $A = \{1, \{1, \{1\}\}\}$ . List all the elements of  $A \times A$ .
- 26. Find all  $z \in \mathbb{C}$  satisfying  $z^2 = |z|^2$ .
- 27. Find the complete integer solution to 28x + 60y = 10.
- 28. Prove or disprove: A prime number can be formed using each of the digits from 0 to 9 exactly once.
- 29. Let gcd(x, y) = d. Express gcd(18x + 3y, 3x) in terms of d and prove that you are correct.
- 30. In the proof of *Prime Factorization (PF)* given in the course notes, why is it okay to write  $r \leq s$ ?
- 31. Let  $a, b, c \in \mathbb{Z}$ . Disprove the statement: If  $a \mid (bc)$ , then  $a \mid b$  or  $a \mid c$ .
- 32. The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by  $\lfloor x \rfloor$ . The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by  $\lceil x \rceil$ . Prove that there is a unique real number x such that  $\lfloor x \rfloor = \lceil x \rceil = 7$ .
- 33. In a strange country, there are only 4 cent and 7 cent coins. Prove that any integer amount of currency greater than 17 cents can always be formed.
- 34. Prove that there exists a polynomial in  $\mathbb{Q}[x]$  with the root  $2 \sqrt{7}$ .
- 35. What is the remainder when  $3141^{2001}$  is divided by 17?
- 36. Suppose that p is a prime and  $a \in \mathbb{Z}$ . Prove using induction that  $a^{(p^n)} \equiv a \pmod{p}$  for all  $n \in \mathbb{N}$ .
- 37. Prove that if gcd(a, b) = 1, then  $gcd(2a + b, a + 2b) \in \{1, 3\}$ .

- 38. How many positive divisors does 33480 have?
- 39. Prove that if |z| = 1 or |w| = 1 and  $\overline{z}w \neq 1$ , then  $\left|\frac{z-w}{1-\overline{z}w}\right| = 1$ .
- 40. Let  $a, b, c \in \mathbb{Z}$ . Consider the statement S: If gcd(a, b) = 1 and  $c \mid (a + b)$ , then gcd(a, c) = 1. Fill in the blanks to complete a proof of S.
  - (a) Since gcd(a, b) = 1, by \_\_\_\_\_\_ there exist integers x and y such that ax + by = 1.
  - (b) Since  $c \mid (a+b)$ , by \_\_\_\_\_\_ there exists an integer k such that a+b=ck.
  - (c) Substituting a = ck b into the first equation, we get 1 = (ck b)x + by = b(-x + y) + c(kx).
  - (d) Since 1 is a common divisor of b and c and -x + y and kx are integers, gcd(b,c) = 1 by
- 41. Express the following complex numbers in standard form.

(a) 
$$\frac{(\sqrt{2}-i)^2}{(\sqrt{2}+i)(1-\sqrt{2}i)}$$
  
(b)  $(\sqrt{5}-i\sqrt{3})^4$ 

- 42. For each of the following polynomials  $f(x) \in \mathbb{F}[x]$ , factor f(x) into factors with degree as small as possible over  $\mathbb{F}[x]$ . Cite appropriate propositions to justify each step of your reasoning.
  - (a)  $x^2 2x + 2 \in \mathbb{C}[x]$ (b)  $x^2 + (-3i+2)x - 6i \in \mathbb{C}[x]$ (c)  $2x^3 - 3x^2 + 2x + 2 \in \mathbb{R}[x]$ (d)  $3x^4 + 13x^3 + 16x^2 + 7x + 1 \in \mathbb{R}[x]$ (e)  $x^4 + 27x \in \mathbb{C}[x]$

43. Let x and y be integers. Prove or disprove each of the following statements.

- (a) If  $2 \nmid xy$  then  $2 \nmid x$  and  $2 \nmid y$ .
- (b) If  $2 \nmid y$  and  $2 \nmid x$  then  $2 \nmid xy$ .
- (c) If  $10 \nmid xy$  then  $10 \nmid x$  and  $10 \nmid y$ .
- (d) If  $10 \nmid x$  and  $10 \nmid y$  then  $10 \nmid xy$ .
- 44. Let S and T be any two sets in universe  $\mathcal{U}$ . Prove that  $(S \cup T) (S \cap T) = (S T) \cup (T S)$ .
- 45. Compute all the fifth roots of unity and plot them in the complex plane.
- 46. Solve  $x^3 \equiv 17 \pmod{99}$ .
- 47. Prove that a prime p divides  $ab^p ba^p$  for all integers a and b.
- 48. Prove that the converse of Divisibility of Integer Combinations (DIC) is true.
- 49. The Chinese Remainder Theorem deals with the case where the moduli are coprime. We now investigate what happens if the moduli are not coprime.
  - (a) Consider the following two systems of linear congruences:

$$A: \left\{ \begin{array}{ll} n \equiv 2 \pmod{12} \\ n \equiv 10 \pmod{18} \end{array} \right. \qquad B: \left\{ \begin{array}{ll} n \equiv 5 \pmod{12} \\ n \equiv 11 \pmod{18} \end{array} \right.$$

Determine which one has solutions and which one has no solutions. For the one with solutions, give the complete solutions to the system. For the one with no solutions, explain why no solutions exist.

(b) Let  $a_1, a_2$  be integers, and let  $m_1, m_2$  be positive integers. Consider the following system of linear congruences

$$S: \left\{ \begin{array}{rrr} n \equiv & a_1 \pmod{m_1} \\ n \equiv & a_2 \pmod{m_2} \end{array} \right.$$

Using your observations in (a), complete the following two statements. The system S has a solution if and only if \_\_\_\_\_\_. If  $n_0$  is a solution to S, then the complete solution is

 $n \equiv$ \_\_\_\_\_

- (c) Prove the first statement.
- 50. Prove the following statements by strong induction.
  - (a) A sequence  $\{x_n\}$  is defined recursively by  $x_1 = 8$ ,  $x_2 = 32$  and  $x_i = 2x_{i-1} + 3x_{i-2}$  for  $i \ge 3$ . For all  $n \in \mathbb{N}$ ,  $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$ .
  - (b) A sequence  $\{t_n\}$  is defined recursively by  $t_n = 2t_{n-1} + n$  for all integers n > 1. The first term is  $t_1 = 2$ . For all  $n \in \mathbb{N}$ ,  $t_n = 5 \times 2^{n-1} 2 n$ .
- 51. Find all complex numbers z solutions to  $z^2 = \frac{1+i}{1-i}$ .
- 52. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.
- 53. Prove that for distinct primes p and q,  $(p^{q-1} + q^{p-1}) \equiv 1 \pmod{pq}$ .
- 54. Set up an RSA scheme using two-digit prime numbers. Select values for the other variables and test encrypting and decrypting messages.
- 55. Let u and v be fixed complex numbers. Let  $\omega$  be a non-real cube root of unity. For each  $k \in \mathbb{Z}$ , define  $y_k \in \mathbb{C}$  by the formula

$$y_k = \omega^k u + \omega^{-k} v.$$

- (a) Compute  $y_1$ ,  $y_2$  and  $y_3$  in terms of u, v and  $\omega$ .
- (b) Show that  $y_k = y_{k+3}$  for any  $k \in \mathbb{Z}$ .
- (c) Show for any  $k \in \mathbb{Z}$ ,

$$y_k - y_{k+1} = \omega^k (1 - \omega)(u - \omega^{k-1}v)$$

56. Prove that: if  $a \mid c$  and  $b \mid c$  and gcd(a, b) = 1, then  $ab \mid c$ .

57. If a and b are integers,  $3 \nmid a$ ,  $3 \nmid b$ ,  $5 \nmid a$ , and  $5 \nmid b$ , prove that  $a^4 \equiv b^4 \pmod{15}$ .

58. Prove the following statements by simple induction.

(a) For all 
$$n \in \mathbb{N}$$
,  $\sum_{i=1}^{n} (2i-1) = n^2$ .  
(b) For all  $n \in \mathbb{N}$ ,  $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$  where  $r$  is any real number such that  $r \neq 1$ .  
(c) For all  $n \in \mathbb{N}$ ,  $\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$ .

- (d) For all  $n \in \mathbb{N}$ ,  $\sum_{i=1}^{n} \frac{i}{2^{i}} = 2 \frac{n+2}{2^{n}}$ .
- (e) For all  $n \in \mathbb{N}$  where  $n \ge 4$ ,  $n! > n^2$ .
- 59. A basket contains a number of eggs and, when the eggs are removed 2, 3, 4, 5 and 6 at a time, there are 1, 2, 3, 4 and 5 respectively, left over. When the eggs are removed 7 at a time there are none left over. Assuming none of the eggs broke during the preceding operations, determine the minimum number of eggs that were in the basket.
- 60. Prove there is a unique set T such that for every set  $S, S \cup T = S$ .
- 61. Let  $a, b, c \in \mathbb{C}$ . Prove: if |a| = |b| = |c| = 1, then  $\overline{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ .
- 62. Prove that if p is prime and  $p \leq n$ , then p does not divide n! + 1.
- 63. Prove that if gcd(a,b) = 1, then  $gcd(a^m,b^n) = 1$  for all  $m, n \in \mathbb{N}$ . You may use the result of an example in the notes.
- 64. Solve  $49x^{177} + 37x^{26} + 3x^2 + x + 1 \equiv 0 \pmod{7}$ .
- 65. Prove that  $\forall z, w \in \mathbb{C}, |z w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$  (This is the Parallelogram Identity).
- 66. Given a rational number r, prove that there exist coprime integers p and q, with  $q \neq 0$ , so that  $r = \frac{p}{q}$ .
- 67. (a) Find all  $w \in \mathbb{C}$  satisfying  $w^2 = -15 + 8i$ ,
  - (b) Find all  $z \in \mathbb{C}$  satisfying  $z^2 (3+2i)z + 5 + i = 0$ .
- 68. Consider the following proposition about integers a and b.

If  $a^3 \mid b^3$ , then  $a \mid b$ .

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

- (a) Consider a = 2, b = 4. Then  $a^3 = 8$  and  $b^3 = 64$ . We see that  $a^3 \mid b^3$  since  $8 \mid 64$ . Since  $2 \mid 4$ , we have  $a \mid b$ .
- (b) Since  $a \mid b$ , there exists  $k \in \mathbb{Z}$  such that b = ka. By cubing both sides, we get  $b^3 = k^3 a^3$ . Since  $k^3 \in \mathbb{Z}, a^3 \mid b^3$ .
- (c) Since  $a^3 \mid b^3$ , there exists  $k \in \mathbb{Z}$  such that  $b^3 = ka^3$ . Then  $b = (ka^2/b^2)a$ , hence  $a \mid b$ .
- 69. Is  $27^{129} + 61^{40}$  is divisible by 14? Show and justify your work.
- 70. Prove that if w is an  $n^{th}$  root of unity, then  $\frac{1}{w}$  is also an  $n^{th}$  root of unity.
- 71. What are the last two digits of  $43^{201}$ ?
- 72. Let x and y be integers. Prove that if xy = 0 then x = 0 or y = 0.
- 73. Let  $z, w \in \mathbb{C}$ . Prove that if zw = 0 then z = 0 or w = 0.
- 74. Determine all  $k \in \mathbb{N}$  such that  $n^k \equiv n \pmod{7}$  for all integers n. Prove that your answer is correct
- 75. Express  $\frac{2-i}{3+4i}$  in standard form.

76. Suppose  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ . We say f(x) is palindromic if the coefficients  $a_j$  satisfy

$$a_{n-j} = a_j$$
 for all  $0 \le j \le n$ .

Prove that

- (a) If f(x) is a palindromic polynomial and  $c \in \mathbb{C}$  is a root of f(x), then c must be non-zero, and  $\frac{1}{c}$  is also a root of f(x).
- (b) If f(x) is a palindromic polynomial of odd degree, then f(-1) = 0.
- (c) If  $\deg(f) = 1$  and f(x) is a monic, palindromic polynomial, then f(x) = x + 1.

77. Let a and b be two integers. Prove each of the following statements about a and b.

- (a) If ab = 4, then  $(a b)^3 9(a b) = 0$ .
- (b) If a and b are positive, then  $a^2(b+1) + b^2(a+1) \ge 4ab$ .
- 78. Give an example of three sets A, B, and C such that  $B \neq C$  and B A = C A.
- 79. What is the remainder when -98 is divided by 7?
- 80. Let n be an integer. Prove that if  $1 n^2 > 0$ , then 3n 2 is an even integer.
- 81. Let  $g(x) = x^3 + bx^2 + cx + d \in \mathbb{C}[x]$  be a cubic polynomial whose leading coefficient is 1 (such polynomials are called *monic*). Let  $z_1, z_2, z_3$  be three roots of g(x), such that

$$g(x) = (x - z_1)(x - z_2)(x - z_3).$$

Prove that

$$z_1 + z_2 + z_3 = -b,$$
  

$$z_1 z_2 + z_2 z_3 + z_3 z_1 = c,$$
  

$$z_1 z_2 z_3 = -d.$$

- 82. How many integers x where  $0 \le x < 1000$  satisfy  $42x \equiv 105 \pmod{56}$ ?
- 83. Prove or disprove: If  $7a^2 = b^2$  where  $a, b \in \mathbb{Z}$ , then 7 is a common divisor of a and b.
- 84. What is the smallest non-negative integer x such that  $2000 \equiv x \pmod{37}$ ?
- 85. A complex number z is called a *primitive* n-th root of unity if  $z^n = 1$  and  $z^k \neq 1$  for all  $1 \leq k \leq n-1$ .
  - (a) For each n = 1, 2, 3, 6, list all the primitive *n*-th roots of unity.
  - (b) Let z be a primitive n-th root of unity. Prove the following statements.
    - i. For any  $k \in \mathbb{Z}$ ,  $z^k = 1$  if and only if  $n \mid k$ .
    - ii. For any  $m \in \mathbb{Z}$ , if gcd(m, n) = 1, then  $z^m$  is a primitive *n*-th root of unity.
- 86. Prove the following two quantified statements.
  - (a)  $\forall n \in \mathbb{N}, n+1 \ge 2$
  - (b)  $\exists n \in \mathbb{Z}, \frac{(5n-6)}{3} \in \mathbb{Z}$
- 87. Solve  $x^3 29x^2 + 35x + 38 \equiv 0 \pmod{195}$ .
- 88. Prove that  $x^2 + 9 \ge 6x$  for all real numbers x.

- 89. Prove that if k is an odd integer, then 4k + 7 is an odd integer.
- 90. Suppose p is a prime greater than five. Prove that the positive integer consisting of p-1 digits all equal to one (111...1) is divisible by p. (Hint:  $111111 = \frac{10^6-1}{9}$ .)
- 91. Let x be a real number. Prove that if  $x^3 5x^2 + 3x \neq 15$  then  $x \neq 5$ .
- 92. Let  $z \in \mathbb{C}$ . Prove that  $(x z)(x \overline{z}) \in \mathbb{R}[x]$ .
- 93. Write  $z = \frac{9+i}{5-4i}$  in the form  $r(\cos \theta + i \sin \theta)$  with  $r \ge 0$  and  $0 \le \theta < 2\pi$ .
- 94. Complete a multiplication table for  $\mathbb{Z}_5$ .
- 95. Suppose a and b are integers. Prove that  $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}.$

96. Evaluate 
$$\sum_{i=3}^{8} 2^i$$
 and  $\prod_{j=1}^{5} \frac{j}{3}$ .

97. Find the complete integer solution to 7x + 11y = 3.

98. Solve

$$3x - 2 \equiv 7 \pmod{11}$$
$$5 \equiv 4x - 1 \pmod{9}$$

- 99. Prove the properties of complex arithmetic given in Proposition 1 in Chapter 30 of the course notes. Only one of the nine results is proved in the notes. A few others may have been proved in class.
- 100. Assume that it has been established that the following implication is true:

If I don't see my advisor today, then I will see her tomorrow.

For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

- (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)
- (b) I meet my advisor both today and tomorrow.
- (c) I meet my advisor either today or tomorrow (but not on both days).
- 101. Which elements of  $\mathbb{Z}_6$  have multiplicative inverses?
- 102. Prove or disprove each of the following statements involving nested quantifiers.
  - (a) For all  $n \in \mathbb{Z}$ , there exists an integer k > 2 such that  $k \mid (n^3 n)$ .
  - (b) For every positive integer a, there exists an integer b with |b| < a such that b divides a.
  - (c) There exists an integer n such that m(n-3) < 1 for every integer m.
  - (d)  $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0.$
- 103. For each linear congruence, determine the complete solution, if a solution exists.
  - (a)  $3x \equiv 11 \pmod{18}$
  - (b)  $4x \equiv 5 \pmod{21}$

- 104. Is 7386458999999992324343123 divisible by 11? Show and justify your work.
- 105. Let  $n \in \mathbb{N}$  and  $a, b \in \mathbb{R}$ . Show that  $z = (a + bi)^n + (a bi)^n$  is real.
- 106. Show that  $|\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2} |z|$ .
- 107. Disprove the following. Let  $a, b, c \in \mathbb{Z}$ . Then  $gcd(a, b) = gcd(a, c) \cdot gcd(b, c)$ .
- 108. Let  $A = \{n \in \mathbb{Z} : 2 \mid n\}$  and  $B = \{n \in \mathbb{Z} : 4 \mid n\}$ . Prove that  $n \in (A B)$  if and only if n = 2k for some odd integer k.
- 109. Let n be an integer. Prove that  $2 \mid (n^4 3)$  if and only if  $4 \mid (n^2 + 3)$ .
- 110. What are the integer solutions to  $x^2 \equiv 1 \pmod{15}$ ?
- 111. Prove that for all  $a \in \mathbb{Z}$ , gcd(9a + 4, 2a + 1) = 1
- 112. Let a and b be integers. Prove that  $(a \mid b \land b \mid a) \iff a = \pm b$ .
- 113. Find all  $z \in \mathbb{C}$  satisfying  $|z+1|^2 \leq 3$  and shade the corresponding region in the complex plane.
- 114. Are the following functions onto? Are they 1-1? Justify your answer with a proof.
  - (a)  $f : \mathbb{Z} \to \mathbb{Z}$ , defined by f(n) = 2n + 1.
  - (b)  $f : \mathbb{R} \to \mathbb{R}$ , defined by  $f(x) = x^2 + 4x + 9$ .
  - (c)  $f: (\mathbb{R} \{2\}) \to (\mathbb{R} \{5\})$ , defined by  $f(x) = \frac{5x+1}{x-2}$ .
- 115. Consider the following statement:

Let  $a, b, c \in \mathbb{Z}$ . For every integer  $x_0$ , there exists an integer  $y_0$  such that  $ax_0 + by_0 = c$ .

- (a) Determine conditions on a, b, c such that the statement is true if and only if these conditions hold. State and prove this if and only if statement.
- (b) Carefully write down the negation of the given statement and prove that this negation is true.
- 116. Let  $n \in \mathbb{N}$ . Prove by induction that if  $n \equiv 1 \pmod{4}$ , then  $i^n = i$ .
- 117. For each of the following statements, identify the four parts of the quantified statement (quantifier, variable, domain, and open sentence). Next, express the statement in symbolic form and then write down the negation of the statement (when possible, without using any negative words such as "not" or the  $\neg$  symbol, but negative math symbols like  $\neq$ ,  $\nmid$  are okay).
  - (a) For all real numbers x and y,  $x \neq y$  implies that  $x^2 + y^2 > 0$ .
  - (b) For every even integer a and odd integer b, a rational number c can always be found such that either a < c < b or b < c < a.
  - (c) There is some  $x \in \mathbb{N}$  such that for all  $y \in \mathbb{N}$ ,  $y \mid x$ .
  - (d) There exist sets of integers X, Y such that for all sets of integers  $Z, X \subseteq Z \subseteq Y$ . (You may use  $\mathcal{P}(\mathbb{Z})$  to denote the set of all sets of integers. This is called *power set notation*.)

118. Find all  $z \in \mathbb{C}$  which satisfy

(a)  $z^{2} + 2z + 1 = 0$ , (b)  $z^{2} + 2\overline{z} + 1 = 0$ , (c)  $z^{2} = \frac{1+i}{1-i}$ .

- 119. Prove that an integer is even if and only if its square is an even integer.
- 120. Divide  $f(x) = x^3 + x^2 + x + 1$  by  $g(x) = x^2 + 4x + 3$  to find the quotient q(x) and remainder r(x) that satisfy the requirements of the Division Algorithm for Polynomials (DAP).
- 121. What is the remainder when  $14^{43}$  is divided by 41?
- 122. Determine whether  $A \implies B$  is logically equivalent to  $(\neg A) \lor B$ .
- 123. Use De Moivre's Theorem (DMT) to prove that  $\sin(4\theta) = 4\sin\theta\cos^3\theta 4\sin^3\theta\cos\theta$ .
- 124. Solve

$$x \equiv 7 \pmod{11}$$
$$x \equiv 5 \pmod{12}$$

- 125. (a) Use the Extended Euclidean Algorithm to find three integers x, y and  $d = \gcd(1112, 768)$  such that 1112x + 768y = d.
  - (b) Determine integers s and t such that  $768s 1112t = \gcd(768, -1112)$ .
- 126. Let a, b, c and d be integers. Prove that if  $a \mid b$  and  $b \mid c$  and  $c \mid d$ , then  $a \mid d$ .
- 127. Suppose a, b and n are integers. Prove that  $n \mid \gcd(a, n) \cdot \gcd(b, n)$  if and only if  $n \mid ab$ .
- 128. Let a, b, c and d be positive integers. Suppose  $\frac{a}{b} < \frac{c}{d}$ . Prove that  $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ .
- 129. Find all non-negative integer solutions to 12x + 57y = 423.