MATH 135: Randomized Midterm Practice Problems

These are the warm-up exercises and recommended problems take from the first four extra practice sets presented in random order. The challenge problems have not been included.

1. Prove that if k is an odd integer, then $4k + 7$ is an odd integer.

2. Let a, b, c and d be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$. Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

3. Consider the following statement.

   For all $x \in \mathbb{R}$, if $x^6 + 3x^4 - 3x < 0$, then $0 < x < 1$.

   (a) Rewrite the given statement in symbolic form.
   (b) State the hypothesis of the implication within the given statement.
   (c) State the conclusion of the implication within the given statement.
   (d) State the converse of the implication within the given statement.
   (e) State the contrapositive of the implication within the given statement.
   (f) State the negation of the given statement without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine).
   (g) Prove or disprove the given statement.

4. What is the remainder when -98 is divided by 7?

5. The Fibonacci sequence is defined as the sequence $\{f_n\}$ where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements.

   (a) For $n \geq 2$,
   $$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$$
   (b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$

6. Prove that there is a unique minimum value of $x^2 - 4x + 11$.

7. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

8. Prove the following statements by simple induction.

   (a) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n}(2i-1) = n^2$.
   (b) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ where r is any real number such that $r \neq 1$. .
   (c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.
   (d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.
   (e) For all $n \in \mathbb{N}$ where $n \geq 4$, $n! > n^2$.

9. Prove that an integer is even if and only if its square is an even integer.

10. Assume that it has been established that the following implication is true:

    If I don't see my advisor today, then I will see her tomorrow.

    For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

    (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)
    (b) I meet my advisor both today and tomorrow.
    (c) I meet my advisor either today or tomorrow (but not on both days).

11. Are the following functions onto? Are they 1-1? Justify your answer with a proof.

    (a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 2n + 1$.
    (b) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 4x + 9$.
    (c) $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$, defined by $f(x) = \frac{5x+1}{x-2}$.

12. Prove that $x^2 + 9 \geq 6x$ for all real numbers x.

13. Let S and T be any two sets in universe $\mathcal{U}$. Prove that $(S \cup T) - (S \cap T) = (S - T) \cup (T - S)$.

14. In the proof by contradiction of *Prime Factorization (PF)*, why is it okay to write $r \leq s$?

15. Evaluate $\displaystyle\sum_{i=3}^{8} 2^i$ and $\displaystyle\prod_{j=1}^{5} \frac{j}{3}$.

16. Let n be an integer. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

17. Let $A = \{n \in \mathbb{Z} \ : \ 2 \mid n\}$ and $B = \{n \in \mathbb{Z} \ : \ 4 \mid n\}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer k.

18. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

19. Let $A = \{1, \{1, \{1\}\}\}$. List all the elements of $A \times A$.

20. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

21. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.

22. Prove or disprove each of the following statements.

    (a) $\forall\, n \in \mathbb{Z}$, $\frac{(5n-6)}{3}$ is an integer.
    (b) For every prime number p, $p + 7$ is composite.
    (c) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.
    (d) The equation $x^3 + x^2 - 1 = 0$ has a real number solution between $x = 0$ and $x = 1$ (inclusive).

23. Prove that there is a unique real number such that $x^2 - 6x + 9 = 0$.

24. Suppose r is some (unknown) real number, where $r \neq -1$ and $r \neq -2$. Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r\,(2^r)}{(r+1)(r+2)}.$$

25. Prove the following two quantified statements.

    (a) $\forall\, n \in \mathbb{N},\ n+1 \geq 2$

    (b) $\exists\, n \in \mathbb{Z},\ \frac{(5n-6)}{3} \in \mathbb{Z}$

26. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

    (a) A sequence $\{x_n\}$ is defined by $x_1 = 3$, $x_2 = 20$ and $x_i = 5x_{i-1}$ for $i \geq 3$. Then, for all $n \in \mathbb{N}$, $x_n = 3 \times 5^{n-1}$.

    Let $P(n)$ be the statement: $x_n = 3 \times 5^{n-1}$.
    When $n = 1$ we have $3 \times 5^0 = 3 = x_1$ so $P(1)$ is true. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $x_k = 3 \times 5^{k-1}$ for some integer $k \geq 1$. We must show that $P(k+1)$ is true, that is, $x_{k+1} = 3 \times 5^k$. Now

    $$x_{k+1} = 5x_k = 5(3 \times 5^{k-1}) = 3 \times 5^k$$

    as required. Since the result is true for $n = k+1$, and so holds for all n by the Principle of Mathematical Induction.

    (b) For all $n \in \mathbb{N}$, $1^{n-1} = 2^{n-1}$.

    Let $P(n)$ be the statement: $1^{n-1} = 2^{n-1}$.
    When $n = 1$ we have $1^0 = 1 = 2^0$ so $P(1)$ is true. Assume that $P(i)$ is true for all integers $1 \leq i \leq k$. That is, $1^{i-1} = 2^{i-1}$ for all $1 \leq i \leq k$.
    We must show that $P(k+1)$ is true, that is, $1^{(k+1)-1} = 2^{(k+1)-1}$ or $1^k = 2^k$. By our inductive hypothesis, $P(2)$ is true so $1^1 = 2^1$. Also by our inductive hypothesis, $P(k)$ is true so $1^{k-1} = 2^{k-1}$. Multiplying these two equations together gives $1^k = 2^k$. Since the result is true for $n = k+1$, and so holds for all n by the Principle of Strong Induction.

27. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$ then $x \neq 5$.

28. Prove that the product of any four consecutive integers is one less than a perfect square.

29. Let a and b be two integers. Prove each of the following statements about a and b.

    (a) If $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$.

    (b) If a and b are positive, then $a^2(b+1) + b^2(a+1) \geq 4ab$.

30. Let x and y be integers. Prove or disprove each of the following statements.

    (a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.

    (b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.

    (c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

    (d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

31. Let n, a and b be positive integers. Negate the following implication without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine). *Implication:* If $a^3 \mid b^3$, then $a \mid b$.

32. Prove the following statements.

    (a) There is no smallest positive real number.
    (b) For every even integer n, n cannot be expressed as the sum of three odd integers.
    (c) If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.
    (d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2+3y^2 = m$.
    (e) The sum of a rational number and an irrational number is irrational.
    (f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

33. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

34. For each of the following statements, identify the four parts of the quantified statement (quantifier, variable, domain, and open sentence). Next, express the statement in symbolic form and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq, \nmid$ are okay).

    (a) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.
    (b) For every even integer a and odd integer b, a rational number c can always be found such that either $a < c < b$ or $b < c < a$.
    (c) There is some $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \mid x$.
    (d) There exist sets of integers X, Y such that for all sets of integers Z, $X \subseteq Z \subseteq Y$.
    (You may use $\mathcal{P}(\mathbb{Z})$ to denote the set of all sets of integers. This is called *power set notation.*)

35. Give an example of three sets A, B, and C such that $B \neq C$ and $B - A = C - A$.

36. Determine whether $A \implies \neg B$ is logically equivalent to $\neg(A \implies B)$.

37. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

38. Let a, b, c be integers. Prove that if $a \mid b$ then $ac \mid bc$.

39. Let a and b be integers. Prove that $(a \mid b \ \wedge \ b \mid a) \iff a = \pm b$.

40. Prove or disprove each of the following statements involving nested quantifiers.

    (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.
    (b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.
    (c) There exists an integer n such that $m(n-3) < 1$ for every integer m.
    (d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$.

41. Prove that the converse of *Divisibility of Integer Combinations (DIC)* is true.

42. Let $a, b, c \in \mathbb{Z}$. Is the following statement true? Prove that your answer is correct.

    $a \mid b$ if and only if $ac \mid bc$.

43. Consider the following proposition about integers a and b.

    If $a^3 \mid b^3$, then $a \mid b$.

    We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

(a) Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.

(b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, $a^3 \mid b^3$.

(c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

44. Suppose S and T are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?

45. Prove the following statements by strong induction.

(a) A sequence $\{x_n\}$ is defined recursively by $x_1 = 8$, $x_2 = 32$ and $x_i = 2x_{i-1} + 3x_{i-2}$ for $i \geq 3$. For all $n \in \mathbb{N}$, $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$.

(b) A sequence $\{t_n\}$ is defined recursively by $t_n = 2t_{n-1} + n$ for all integers $n > 1$. The first term is $t_1 = 2$. For all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.