Lecture 9

Handout or Document Camera or Class Exercise

Rewrite the following using as few English words as possible.

(i) No multiple of 15 plus any multiple of 6 equals 100.

(ii) Whenever three divides both the sum and difference of two integers, it also divides each of these integers.

Solution:

(i) $\forall m, n \in \mathbb{Z}, (15m + 6n \neq 100)$

(ii) $\forall m, n \in \mathbb{Z}, ((3 \mid (m + n) \wedge 3 \mid (m - n)) \Rightarrow 3 \mid m \wedge 3 \mid n)$

Instructor's Comments: This is the 10 minute mark

Handout or Document Camera or Class Exercise

Write the following statements in (mostly) plain English.

(i) $\forall m \in \mathbb{Z}, ((\exists k \in \mathbb{Z}, m = 2k) \Rightarrow (\exists \ell \in \mathbb{Z}, 7m^2 + 4 = 2\ell))$

(ii) $n \in \mathbb{Z} \Rightarrow (\exists m \in \mathbb{Z}, m > n)$

Solution:

(i) If m is an even integer, then $7m^2 + 4$ is even.

(ii) There is no greatest integer. (Alternatively, for every integer, there exists a greater integer).

Instructor's Comments: This is the 20 minute mark

Contrapositive

Note: Proofs are not always easy to discover. Sometimes you can convert a given problem to an easier equivalent problem.

Example: $7 \nmid n \Rightarrow 14 \nmid n \equiv 14 \mid n \Rightarrow 7 \mid n$

Definition: The *contrapositive* of $H \Rightarrow C$ is $\neg C \Rightarrow \neg H$.

Note: $H \Rightarrow C \equiv \neg C \Rightarrow \neg H$. This follows since

$$\begin{aligned} H \Rightarrow C &\equiv \neg H \vee C \\ &\equiv C \vee \neg H \\ &\equiv \neg(\neg C) \vee \neg H \\ &\equiv \neg C \Rightarrow \neg H \end{aligned}$$

or by using a Truth table

H	C	$H \Rightarrow C$	$\neg C$	$\neg H$	$\neg C \Rightarrow \neg H$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since the third and sixth columns are equal, their headings are logically equivalent.

Instructor's Comments: This is the 32-37 minute mark

Example: Let $x \in \mathbb{R}$. Prove $x^3 - 5x^2 + 3x \neq 15 \Rightarrow x \neq 5$.

Proof: We prove the contrapositive. Let $x = 5$. Then

$$\begin{aligned} x^3 - 5x^2 + 3x &= (5)^3 - 5(5)^2 + 3(5) \\ &= 5^3 - 5^3 + 15 \\ &= 15. \end{aligned}$$

$\blacksquare$

Example: Suppose $a, b \in \mathbb{R}$ and $ab \in \mathbb{R} - \mathbb{Q}$ (the set of irrational numbers). Show either $a \in \mathbb{R} - \mathbb{Q}$ or $b \in \mathbb{R} - \mathbb{Q}$.

Proof: Proceed by the contrapositive. Suppose that a is rational and b is rational. Then $\exists k, \ell, m, n \in \mathbb{Z}$ such that $a = \frac{k}{\ell}$ and $b = \frac{m}{n}$ with $\ell, n \neq 0$. Then

$$ab = \frac{km}{\ell n} \in \mathbb{Q}$$

as required. $\blacksquare$

Instructor's Comments: This is the 50 minute mark.