Quantified Statements.

$\boxed{1}$ For every natural number n, $2n^2+11n+15$ is composite.

$\boxed{2}$ There is an integer k such that $6=3k$

$\forall$: For all symbol.

$\boxed{1}$ $\forall n \in \mathbb{N}$, $2n^2+11n+15$ is composite.

$\boxed{2}$ $\exists k \in \mathbb{Z}$ s.t. $6=3k$.

Quantifiers Variables domains Open sentence (involving the variable)

$\forall x \in S$, $P(x)$: For all x in S, statement $P(x)$ holds

$x \in S \Rightarrow P(x)$

Proof $\boxed{1}$. Let n be an arbitrary natural number. Then factoring gives $2n^2+11n+15=(2n+5)(n+3)$
Since $2n+5>1$ and $n+3>1$, we have $2n^2+11n+15$ is composite.

$\exists k \in \mathbb{Z}$ s.t. $6 = 3k$

Pf of $\boxed{2}$ Since $3 \cdot 2 = 6$, $k = 2$ satisfies the statement.

Ex: $S \subseteq T \equiv \forall x \in S, x \in T.$

Prove there is an $x \in \mathbb{R}$ such that $\frac{x^2+3x-3}{2x+3} = 1$.

When $x=2$, note $\dfrac{2^2+3(2)-3}{2(2)+3} = \dfrac{7}{7} = 1$. ◭

$\dfrac{x^2+3x-3}{2x+3} = 1 \iff x^2+3x-3 = 2x+3 \iff x$

(PROVIDED $x \neq -\frac{3}{2}$)

Note: Vacuously true statement(s)

$\forall x \in \emptyset, P(x)$.

Ex: Let $a,b,c \in \mathbb{Z}$. If $\forall x \in \mathbb{Z}, a \mid (bx+c)$ then $a \mid (b+c)$.

Pf: Assume $\forall x \in \mathbb{Z}, a \mid (bx+c)$. For example, when $x=1$, $a \mid (b(1)+c)$. Thus $a \mid (b+c)$ $\blacksquare$.

Q: $\exists\, m \in \mathbb{Z}$ s.t. $\frac{m-7}{2m+4} = 5$.

A: When $m=-3$, note $\frac{m-7}{2m+4} = \frac{-3-7}{2(-3)+4} = \frac{-10}{-2} = 5$ $\blacksquare$

Show that for each $x \in \mathbb{R}$, $x^2 + 4x + 7 > 0$.

Let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} x^2+4x+7 &= x^2+4x+4-4+7 \\ &= (x+2)^2+3 \\ &> 0 \quad . \qquad \boxtimes . \end{aligned}$$

Sometimes $\forall$ and $\exists$ are hidden! If you encounter a statement with quantifiers, take a moment to make sure you understand what the question is saying/asking.

Examples:

1. $2n^2 + 11n + 15$ is never prime when n is a natural number. $\forall n \in \mathbb{N},\ 2n^2+11n+15$ is not prime.
2. If n is a natural number, then $2n^2 + 11n + 15$ is composite. $\forall n \in \mathbb{N},\ 2n^2+11n+15$ is composite.
3. $\frac{m-7}{2m+4} = 5$ for some integer m. $\exists m$ s.t. $\frac{m-7}{2m+4} = 5$.
4. $\frac{m-7}{2m+4} = 5$ has an integer solution.

Domain is Important!

Let $P(x)$ be the statement $x^2 = 2$.

Let $S = \{-\sqrt{2}, \sqrt{2}\}$.

Which of the following are true?

$\exists x \in \mathbb{Z}, P(x)$ FALSE $\quad$ $\forall x \in \mathbb{Z}, P(x)$ FALSE.

$\exists x \in \mathbb{R}, P(x)$ TRUE $\quad$ $\forall x \in \mathbb{R}, P(x)$ FALSE.

$\exists x \in S, P(x)$ TRUE. $\quad$ $\forall x \in S, P(x)$ TRUE.