To Infinity and Beyond

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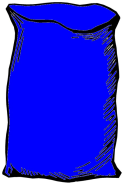
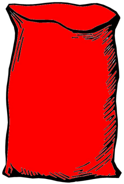
How do we count things?

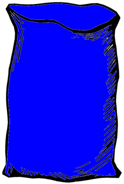
Suppose we have two bags filled with candy. In one bag we have blue candy and in the other bag we have red candy. How can we determine which bag has more candy?

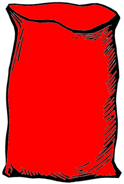




Count Without Counting

What if we don't want to actually count them? Can we determine which bag has more candy without counting the number of each type of candy in each bag?

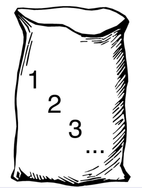




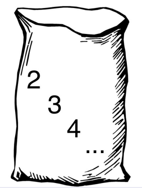
How do we count things?

Now, keeping with the theme, let's replace the candies with numbers. Consider the following two bags of numbers. Which has more?

$S = \{1, 2, 3, ...\}$



$T = \{2, 3, 4, ...\}$



Bijections

Definition 1.

Let S, T be two sets. A function $f : S \to T$ is said to be a bijection if it is one to one and onto. Two sets are said to be bijective if there is a bijection between the two sets. In this case, we say that the sets S and T have the same cardinality and we write $|S| = |T|$ or $\#S = \#T$.

An Example

Always a Bijection

A set S is always bijective with itself by taking the function $f : S \to S$ which sends an element $s \in S$ to itself.

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Another Example

The set $S = \{1, 2, 3, ...\}$ and $T = \{2, 3, 4...\}$ are bijective by taking the function

$$f : S \to T$$
$$s \mapsto s + 1$$

Hotels

Suppose now that you're a big shot manager of a hotel.

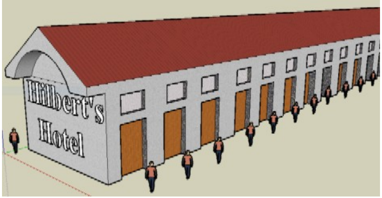
You have been doing very well for yourself and your hotel is fully booked.

Hotels

A person, Georg Cantor, after a long journey, decides that he wants to stay at your hotel. Sadly, you have to turn him away because you have no room.

Hilbert's Hotel

Your competitor, Hilbert's Hotel, owned by David Hilbert, is quite the amazing hotel. At his hotel, he has an infinite number of rooms all connected to a central intercom. Currently Hilbert's Hotel is also full.



Hilbert's Hotel

Georg wants to stay at Hilbert's Hotel. The front desk decides to call David and tell him that a person wants to stay there but the hotel has no vacancies. David, being a good mathematician, says not to worry and that he'll fix the problem.

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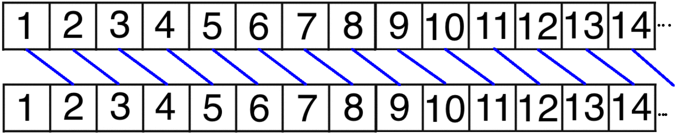
How does David find room for Georg?

Room Shuffle

David decides to call all the rooms at once and tells them to leave their room and go to the next room over.



Hilbert's Hotel Part 2

Georg, amazed that David could accommodate him, decides to call 7 of his friends for a card game and tells them to stay at the hotel overnight. His friends arrive and the front desk informs the group that the hotel is full but that they will call the owner and see what he can do. They call David and once again he manages to find a way to get them to stay at the hotel.

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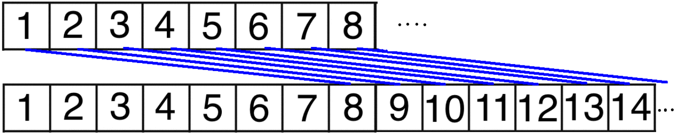
How does Hilbert find room for the group?

Room Shuffle

David calls all the rooms at once and tells each guest to leave their rooms and go to the room that is 7 more than their current room.



Hilbert's Hotel Part 3

Success is never long lived at Hilbert's Hotel. After the group finally settled into their rooms, they decide to call their friends at Infinite Tour Line and tell them to come to Hilbert's Hotel to stay the night. They warned them that the hotel was full but that the manager was an intelligent person and could find a way to stay in.

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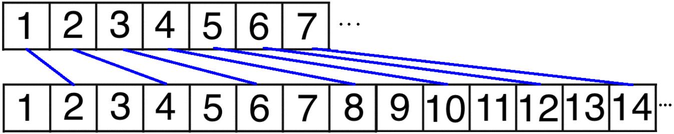
How does Hilbert find room for the Infinite Tour Line?

Room Shuffle

David decides to call all the rooms at once and tells them to leave their room and go to the room that is 2 times more than the current room. He then tells the tour group to one by one go into all the odd numbered rooms.



Hilbert's Hotel Part 4

Infinite Tour Line, amazed at the abilities of David Hilbert, decide to call all of their friends in the tour business and tell them to come to Hilbert's Hotel to stay the night. So the next night, infinitely many busses with infinitely many people in each decided to come to Hilbert's Hotel to spend the night. The front staff, unable to comprehend the sheer number of people, quickly called David for help.

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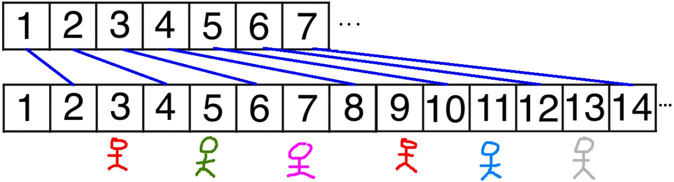
How does Hilbert find room for the infinitely many busses filled with infinitely many people?

Room Shuffle

Hilbert decides to call all the rooms at once and tells them to leave their room and go to the room that is 2 times more than the current room. Then David tells the first bus to go to the rooms that are powers of 3 (namely 3, 9, 27,...,). He then tells the second bus to go into the powers of 5 (namely 5, 25, 125, ...) and so on. David even managed to keep room infinitely many rooms free (1,15,21,...)!



No Vacancies Guests Welcome!

(Courtesy of The Open University open.edu/youtube)

The Other Side of Hilbert's Hotel

The CEO of Infinite Tour Line, Archimedes of Syracus needed to settle up for the evening with Hilbert's Hotel. Zeno of Elea, the CFO of the hotel, seeing as he has infinitely many customers had no worry about money and told Archimedes to set his price for his tour group so long as every person paid a non-zero price.

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What pricing scheme did Archimedes use for his tour group?

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Claim: A person traveling from A to B will never reach B.

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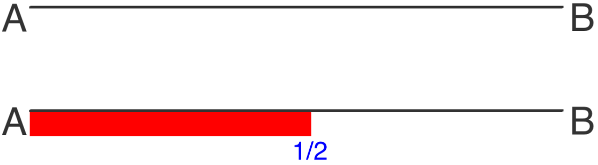

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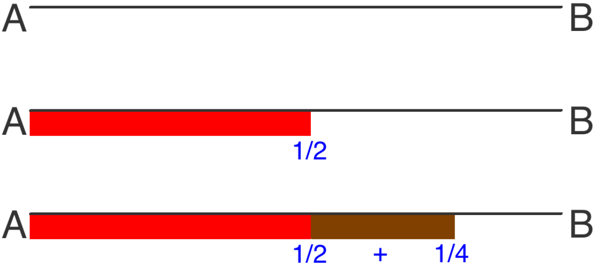

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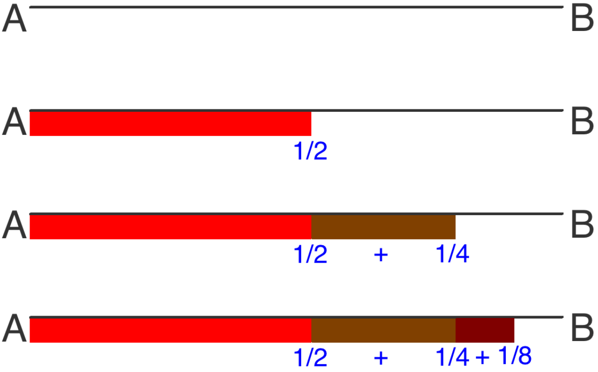

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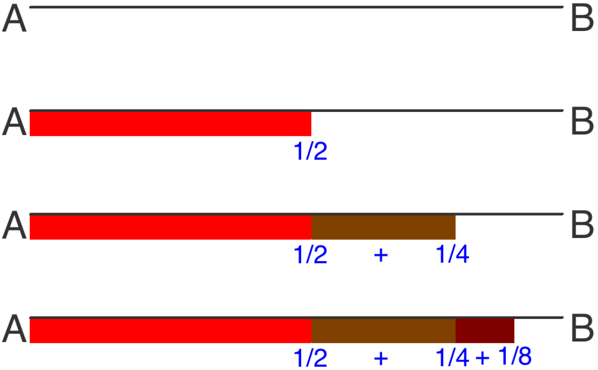

... and so on. Zeno believed when you added infinitely many numbers you got infinity.

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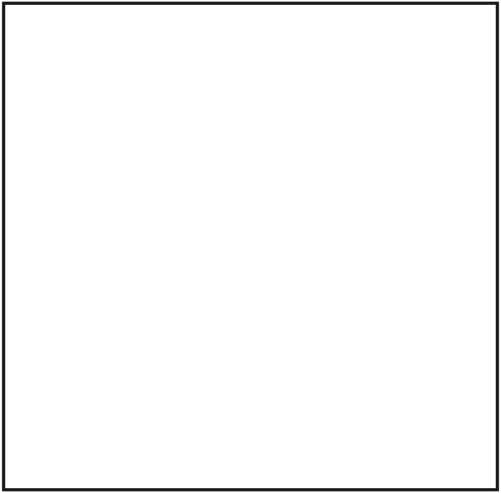
- Cannot just add forever.
- Calculus gives us an explanation using partial summations and limits (which I won't spoil here!)
- A picture here however gives us reasonable insight into how to add this up:

A Finite Infinite Sum

Consider a square with area (and side length) 1.



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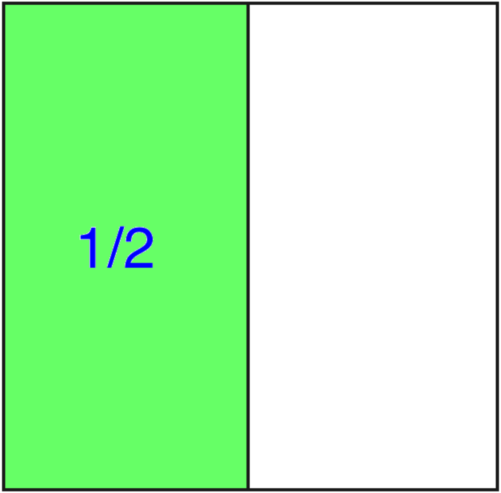

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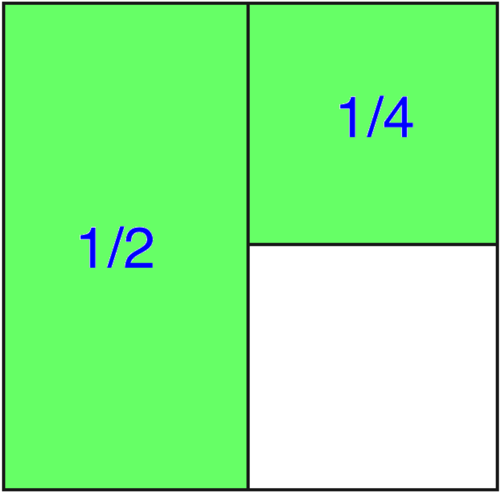

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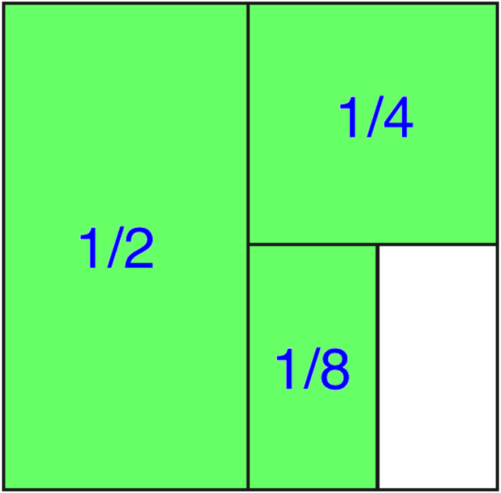

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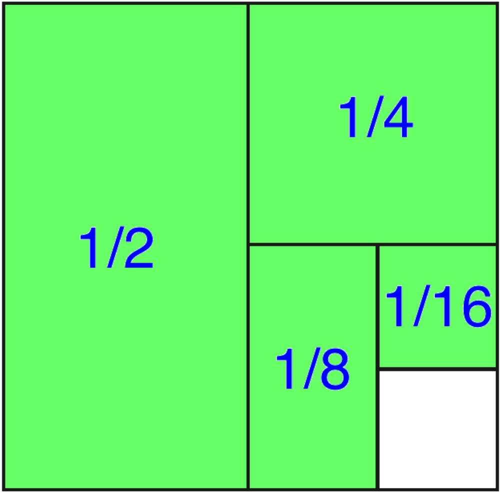

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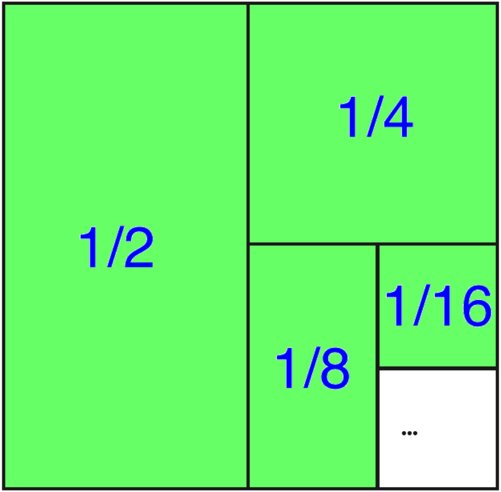

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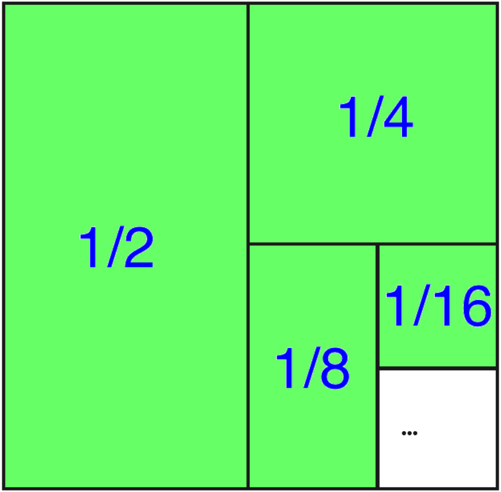

$$1/2 + 1/4 + 1/8 + 1/16 + \dots = 1$$

You're Fired!

David, seeing no choice, fired Zeno right after the transaction closed and decided to replace him with a new CFO, Nicholas Oresme. The Infinite Tour Line, seeing the great deal decided to stay at the hotel again the next week. This time, Oresme decided to charge them 1 of a dollar for the first person, $1/2$ of a dollar for the second person, $1/3$ of a dollar for the third person and in general, $1/n$ dollars for the nth person. Archimedes, seeing that he got the better of Hilbert's Hotel last time, decided that he would agree with this scheme.

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How much money did Archimedes end up paying?

Early Retirement!

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + \ldots$$

Early Retirement!

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + \ldots$$
$$= 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots$$

Early Retirement!

$$\begin{aligned} &1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + \ldots \\ &= 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \\ &> 1 + 1/2 + (1/4 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \end{aligned}$$

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Adding infinitely many of the same term becomes an infinitely large number!

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- At step n, after $1/2^n$ an hour, we put the balls numbered $10n - 9$ to $10n$ in the same box and remove ball n.

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Now $1/2 + 1/4 + 1/8 + \ldots = 1$ hour so the experiment stops. How many balls are in the box at the end of the experiment?

Multiple Infinities?

Are all notions of infinity the same? Can something be more than infinite? Is it a well defined notion?

Notation

Definition 2.

We write $|S| < |T|$ if there is an injection from S to T but no injection from T to S.

Example

For example, $|\{1,2,3\}| < |\mathbb{N}|$ since $\{1,2,3\}$ injects into $\mathbb{N}$ but the natural numbers to not inject into a finite set.

???

Claim: Let V be the interval $(0, 1)$ in the real numbers. Then $|V| > |\mathbb{N}|$. That is, the cardinality of the real numbers is strictly larger than the cardinality of the numbers.

Cantor's Diagonalization Argument.

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- Assume towards a contradiction that V injects into $\mathbb{N}$, that is, there is a one to one function $f : V \to \mathbb{N}$. Then we can enumerate the real numbers by letting a_1 be the real number such that $f(a_1) = 1$ and in general let a_i be the real number that satisfies $f(a_i) = i$.

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- We will use these numbers to create a new real number in V that differs from each a_i.

Cantor's Diagonalization Argument.

- Let $a_{i,j}$ be the jth digit after the decimal point of the real number a_i. Create a new real number given by $b = 0.b_1 b_2 ...$ such that

$$b_j = \begin{cases} 3 & \text{if } a_{j,j} = 7 \\ 7 & \text{if } a_{j,j} \neq 7 \end{cases}$$

Cantor's Diagonalization Argument.

For example, if

$$a_1 = 0.\mathbf{7}32...$$
$$a_2 = 0.3\mathbf{3}6...$$
$$a_3 = 0.81\mathbf{2}...$$
$$...$$

then b would look like

$$b = 0.377...$$

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- **Proof:** Assume towards a contradiction that $b = a_i$. Then, $b_i = a_{i,i}$. However, if $a_{i,i} = 7$, then by definition $b_i = 3 \neq a_{i,i}$ and if $a_{i,i} \neq 7$ then $b_i = 7 \neq a_{i,i}$. This contradiction shows that $b \neq a_i$ for any $i \in \mathbb{N}$.

Cantor's Diagonalization Argument.

This is called Cantor's Diagonalization Argument, after Georg Cantor, because the digit that differ lies along the diagonal if the real numbers were enumerated via:

$$0.\mathbf{a}_{1,1}a_{1,2}a_{1,3}\ldots$$
$$0.a_{2,1}\mathbf{a}_{2,2}a_{2,3}\ldots$$
$$0.a_{3,1}a_{3,2}\mathbf{a}_{3,3}\ldots$$
$$\ldots$$

Continuum Hypothesis

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- Now, an interesting question is whether there is an infinity between the cardinality of the natural numbers and the cardinality of the real numbers.
- This is known as the Continuum Hypothesis.

Continuum Hypothesis

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Continuum Hypothesis

- It was first stated by Cantor in the 1890s and was listed as one of David Hilbert's 23 open problems in the famous 1900 International Congress of Mathematicians in Paris.
- Work of Kurt Gödel in 1940 and Paul Cohen in 1963 combine to show that the Continuum Hypothesis is independent of the usual ZFC axioms of mathematics, that is, you can neither prove nor disprove the statement using the usual axioms.