# Lecture 44

Handout or Document Camera or Class Exercise

How many of the following statements are true?

- Every complex cubic polynomial has a complex root.
- When $x^3 + 6x - 7$ is divided by a quadratic polynomial $ax^2 + bx + c$ in $\mathbb{R}[x]$, then the remainder has degree 1.
- If $f(x), g(x) \in \mathbb{Q}[x]$, then $f(x)g(x) \in \mathbb{Q}[x]$.
- Every non-constant polynomial in $\mathbb{Z}_5[x]$ has a root in $\mathbb{Z}_5$.

A) 0

B) 1

C) 2

D) 3

E) 4

**Solution:** The first statement is true by the Fundamental Theorem of Algebra. The second is false since $x - 1$ is a factor of the cubic polynomial and so there must be a quadratic factor as well. The third is true since $\mathbb{Q}[x]$ forms a ring. The last is false since say $f(x) = x(x-1)(x-2)(x-3)(x-4) + 1$ has no roots over $\mathbb{Z}_5[x]$. Hence the answer is 2.

Recall:

**Theorem:** (Conjugate Roots Theorem (CJRT)) If $c \in \mathbb{C}$ is a root of a polynomial $p(x) \in \mathbb{R}[x]$ (over $\mathbb{C}$) then $\bar{c}$ is a root of $p(x)$.

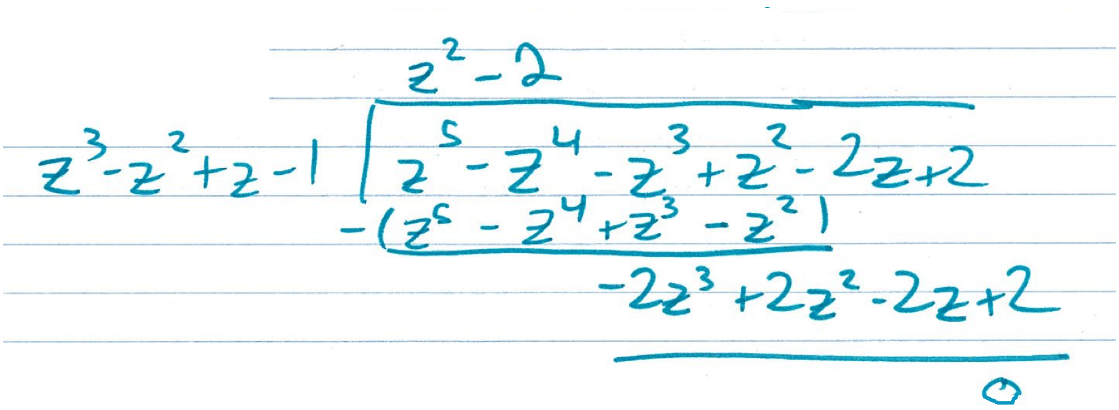
**Note:** This is not true if the coefficients are not real, for example $(x+i)^2 = x^2 + 2ix - 1$.

**Example:** Factor

$$f(z) = z^5 - z^4 - z^3 + z^2 - 2z + 2$$

over $\mathbb{C}$ as a product of irreducible elements of $\mathbb{C}[x]$ given that $i$ is a root.

**Proof:** Note by CJRT that $\pm i$ are roots. By the Factor Theorem, we see that $(z-i)(z+i) = z^2+1$ is a factor. Note that $z-1$ is also a factor since the sum of the coefficients is 0.Hence, $(z^2+1)(z-1) = z^3 - z^2 + z - 1$ is a factor. By long division,




we see that $f(z) = (z^3 - z^2 + z - 1)(z^2 - 2) = (z-i)(z+i)(z-1)(z-\sqrt{2})(z+\sqrt{2})$ is a full factorization. ■

Factor $f(z) = z^4 - 5z^3 + 16z^2 - 9z - 13$ over $\mathbb{C}$ into a product of irreducible polynomials given that $2 - 3i$ is a root.

Factors are (using the Factor Theorem and CJRT)

$$(z - (2 - 3i))(z - (2 + 3i)) = z^2 - 4z + 13$$

After long division,

$$f(z) = (z^2 - 4z + 13)(z^2 - z - 1)$$

By the quadratic formula on the last quadratic,

$$\begin{aligned} z &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Hence, $f(z) = (z - (2 - 3i))(z - (2 + 3i))(z - (1 + \sqrt{5})/2)(z - (1 - \sqrt{5})/2)$. ■

**Theorem:** (Real Quadratic Factors (RQF)) Let $f(x) \in \mathbb{R}[x]$. If $c \in \mathbb{C}-\mathbb{R}$ and $f(c) = 0$, then there exists a $g(x) \in \mathbb{R}[x]$ such that $g(x)$ is a real quadratic factor of $f(x)$.

**Proof:** Let $c \in \mathbb{C}$ be a root of $f(x)$ where $\mathrm{Im}(c) \neq 0$. Then by the Factor Theorem,

$$f(x) = (x - c)q_1(x) \text{ for some } q_1(x) \in \mathbb{C}[x].$$

Now, by the Conjugate Roots Theorem, $\overline{c}$ is also a root of $f(x)$. Hence

$$f(\overline{c}) = (\overline{c} - c)q_1(\overline{c}) = 0.$$

Since $\mathrm{Im}(c) \neq 0$, then $\overline{c} \neq c$, or $\overline{c} - c \neq 0$ which in turn means $q_1(\overline{c}) = 0$. That is, $\overline{c}$ is a root of $q_1(x)$ and so by using the Factor Theorem again, we get that

$$q_1(x) = (x - \overline{c})q_2(x) \text{ where } q_2(x) \in \mathbb{C}[x].$$

We substitute to get

$$f(x) = (x - c)(x - \overline{c})q_2(x) = g(x)q_2(x)$$

where $g(x) = (x - c)(x - \overline{c})$. By Properties of Conjugates and Properties of Modulus,

$$g(x) = x^2 - (c + \overline{c})x + c\overline{c} = x^2 - 2\mathrm{Re}(c)x + |c|^2.$$

Since $-2\mathrm{Re}(c) \in \mathbb{R}$ and $|c|^2 \in \mathbb{R}$, $g(x)$ is a real quadratic polynomial. All that remains is to show that $q_2(x)$ is in $\mathbb{R}[x]$. From above, in $\mathbb{C}[x]$, we have that

$$f(x) = g(x)q_2(x) + r_2(x)$$

where $r_2(x)$ is the zero polynomial. Using the Division Algorithm for Polynomials (DAP) in $\mathbb{R}[x]$, we get

$$f(x) = g(x)q(x) + r(x)$$

where $q(x)$ is in $\mathbb{R}[x]$ and the remainder $r(x)$ is the zero polynomial or $\deg r(x) < \deg g(x)$. Now, every real polynomial is a complex polynomial, so we can also view this as a statement in $\mathbb{C}[x]$. As for any field, DAP over $\mathbb{C}$ tells us that the quotient and remainder are unique. Therefore $r(x) = r_2(x)$ is the zero polynomial and $q(x) = q_2(x)$ has real coefficients.

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Prove that a real polynomial of odd degree has a real root.

**Solution:** Assume towards a contradiction that $p(x)$ is a real polynomial of odd degree without a root. By the Factor Theorem, we know that if $p(x)$ cannot have a real linear factor. By Real Factors of Real Polynomials, we see that

$$p(x) = q_1(x)...q_k(x)$$

for some quadratic factors $q_i(x)$. Now, taking degrees shows that

$$\deg(p(x)) = 2k$$

contradicting the fact that the degree was of $p(x)$ is odd. Hence, the polynomial must have a real root. ■