Recall:

Conjugate Roots Theorem

If $c \in \mathbb{C}$ is a root of a real polynomial, then $\bar{c} \in \mathbb{C}$ is also a root.

Not true if coefficients are not real

Ex: $(x+i)^2 = x^2 + 2ix - 1$

Ex: Fully factor

$$f(z) = z^5 - z^4 - z^3 + z^2 - 2z + 2$$

over $\mathbb{C}$ given that i is a root.

Pf: Note by CJRT $\pm i$ are roots. By FT $(z-i)(z+i) = z^2+1$ is a factor. Note $z-1$ is also a factor hence $(z^2+1)(z-1) = z^3 - z^2 + z - 1$ is a factor.

$$z^2 - 2$$

$$z^3 - z^2 + z - 1 \;\big|\; z^5 - z^4 - z^3 + z^2 - 2z + 2$$

$$-(z^5 - z^4 + z^3 - z^2)$$

$$-2z^3 + 2z^2 - 2z + 2$$

$$\therefore f(z) = (z^3 - z^2 + z - 1)(z^2 - 2)$$
$$= (z-i)(z+i)(z-1)(z-\sqrt{2})(z+\sqrt{2})$$

$\boxtimes$

Fully factor $f(z) = z^4 - 5z^3 + 16z^2 - 9z - 13$ over $\mathbb{C}$ given that $2 - 3i$ is a root.

Factors are (by FT & CJRT)

$$(z-(2-3i))(z-(2+3i))$$
$$= z^2 - 4z + 13$$

After long division

$$f(z) = (z^2 - 4z + 13)(z^2 - z - 1)$$

By the quadratic formula on $z^2 - z - 1$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

Hence $f(z) = (z-(2-3i))(z-(2+3i))$
$$\cdot \left(z - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(z - \left(\frac{1-\sqrt{5}}{2}\right)\right)$$

Real Quadratic Factors (RQF).

Let $f(x) \in \mathbb{R}[x]$. If $c \in \mathbb{C} \setminus \mathbb{R}$ & $f(c)=0$ then $\exists\, g(x) \in \mathbb{R}[x]$ s.t. $g(x)$ is a real quadratic factor of $f(x)$.

Pf: Take
$$g(x) = (x-c)(x-\bar{c})$$
$$= x^2 - (c+\bar{c})x + c\bar{c}$$
$$= x^2 - 2\,\mathrm{Re}(c)\,x + |c|^2 \in \mathbb{R}[x]$$

It suffices to show that $g(x)$ is a factor of $f(x)$. By DAP, $\exists!\, q(x), r(x) \in \mathbb{R}[x]$ s.t.
$$f(x) = g(x)q(x) + r(x) \quad (1).$$
with $r(x) = 0$ OR $\deg(r(x)) < \deg(g(x)) = 2$

Assume towards a contradiction that $r(x) \neq 0$ ie $\deg(r(x)) = 0$ or 1. Plug $x=c$ into (1)
$$0 = f(c) = g(c)q(c) + r(c) = r(c).$$
$$\therefore r(c) = 0.$$

Now, $r(x)$ is linear or constant real polynomial

If $r(x)$ was constant, $r(x)=0$ #

If $r(x)$ is linear, say $r(x)=ax+b$, then

$r(c)=ac+b=0 \Rightarrow c=\frac{-b}{a}\in\mathbb{R}$ #.

$\therefore r(x)=0$ & $g(x)\mid f(x)$ ■.

Real Factors of Real Polynomials (RFPF).

Let $f(x)=a_n x^n+\cdots+a_1x+a_0\in\mathbb{R}[x]$.

Then $f(x)$ can be written as a product of real linear & real quadratic factors.

PF: By CPN, $f(x)$ has n roots over $\mathbb{C}$. Let $r_1, r_2, \dots r_k$ be the real roots and let $c_1, c_2, \dots c_\ell$ be the complex roots. By CJRT complex roots come in pairs say $c_2=\overline{c_1}$, $c_4=\overline{c_3}$, ... $c_\ell=\overline{c_{\ell-1}}$. For each pair, by RQF, we have an associated quadratic factor, say $q_1(x), q_2(x), \dots, q_{\ell/2}(x)$.

By FT, each real root corresponds to a linear factor, say $g_1(x), g_2(x), \dots, g_k(x)$

Then $f(x) = c\, g_1(x) g_2(x) \cdots g_k(x) q_1(x) \cdots q_{\frac{\ell}{2}}(x)$

$\boxed{}$

L44P7

Prove that a real polynomial of odd degree has a real root.