L43P1

Q1. I enjoy trying to discover and write MATH 135 proofs.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q2. When I have difficulties with MATH 135 proofs, I know I can handle them.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q3. How many of the following statements are true?

- TRUE FTA — Every complex cubic polynomial has a complex root.
- FALSE — When $x^3 + 6x - 7$ is divided by $ax^2 + bx + c$ in $\mathbb{R}[x]$, then the remainder has degree 1.
- TRUE — If $f(x), g(x) \in \mathbb{Q}[x]$, then $f(x)g(x) \in \mathbb{Q}[x]$.
- FALSE — Every polynomial in $\mathbb{Z}_5[x]$ has a root in $\mathbb{Z}_5$.

A) 0 $\quad f(x) = 1$

B) 1

$\boxed{\text{C) 2}}$ $\quad f(x) = x(x-1)(x-2)(x-3)(x-4)+1$

D) 3

E) 4

Rational Roots Theorem (RRT)

If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ & $r = s/t \in \mathbb{Q}$ is a root of $f(x)$ over $\mathbb{Q}$ in lowest terms, then $s \mid a_0$ & $t \mid a_n$

Pf: Plug in $r = \frac{s}{t}$ into $f(x)$:

$$0 = a_n \left(\frac{s}{t}\right)^n + a_{n-1}\left(\frac{s}{t}\right)^{n-1} + \cdots + a_1\left(\frac{s}{t}\right) + a_0$$

Multiply by t^n:

$$0 = a_n s^n + a_{n-1} s^{n-1} t + \cdots + a_1 s t^{n-1} + a_0 t^n$$

$$a_0 t^n = -(a_n s^n + a_{n-1} s^{n-1} t + \cdots + a_1 s t^{n-1})$$

$$= -s(a_n s^{n-1} + a_{n-1} s^{n-2} t + \cdots + a_1 t^{n-1})$$

$\therefore s \mid a_0 t^n$. Since $\gcd(s,t) = 1$, $\gcd(s, t^n) = 1$ hence $s \mid a_0$ by CAD.

Similarly $t \mid a_n$ $\blacksquare$.

Ex: Find the roots of

$$2x^3 + x^2 - 6x - 3 \in \mathbb{R}[x]$$

Sol'n: By RRT if r is a root then writing $r = \frac{s}{t}$, we have that $s \mid -3$ & $t \mid 2$

Thus, $r \in \{\pm 1, \pm 3, \pm \frac{3}{2}, \pm \frac{1}{2}\}$

Now, trying these one by one shows that $r = -\frac{1}{2}$ is a root since

$$2(-\tfrac{1}{2})^3 + (-\tfrac{1}{2})^2 - 6(-\tfrac{1}{2}) - 3$$

$$= -\frac{1}{4} + \frac{1}{4} + 3 - 3$$

$$= 0.$$

$\therefore$ $(x + \frac{1}{2})$ or $(2x+1)$ is a factor!

By long division:

$$2x^3 + x^2 - 6x - 3 = (2x+1)(x^2 - 3)$$

$$= (2x+1)(x - \sqrt{3})(x + \sqrt{3})$$

$\therefore$ All real roots are $-\frac{1}{2}, \pm\sqrt{3}$.

Fully factor $x^3 - \frac{32}{15}x^2 + \frac{1}{5}x + \frac{2}{15} \in \mathbb{Q}[x]$

$$= \frac{1}{15}(15x^3 - 32x^2 + 3x + 2) = f(x)$$

By RRT, Possible roots are

$$\pm 1, \pm \frac{1}{3}, \pm \frac{1}{5}, \pm \frac{1}{15}, \pm 2, \pm \frac{2}{3}, \pm \frac{2}{5}, \pm \frac{2}{15}.$$

Note: $x = 2$ is a root. By FT, $x-2$ is a factor.

$$\begin{array}{r|l} & 15x^2 - 2x - 1 \\ \hline x-2 & 15x^3 - 32x^2 + 3x + 2 \\ & 15x^3 - 30x^2 \\ \hline & \quad -2x^2 + 3x \\ & \quad -2x^2 + 4x \\ \hline & \qquad -x + 2 \end{array}$$

$$\therefore f(x) = \frac{1}{15}(x-2)(15x^2 - 2x - 1)$$

$$= \frac{1}{15}(x-2)(5x+1)(3x-1).$$

Prove $\sqrt{7}$ is irrational.

Assume towards a contradiction that

$$\sqrt{7} = x \in \mathbb{Q}$$

Square both sides:

$$7 = x^2$$

$$0 = x^2 - 7$$

As a polynomial, $x^2 - 7$ has a rational root. By RRT, the only possible rational roots are given by $\pm 1, \pm 7$. None of these are roots. (Check!) $\blacksquare$.

$(\pm 1)^2 - 7 = -6 \neq 0 \qquad (\pm 7)^2 - 7 = 42 \neq 0.$

Prove that $\sqrt{5} + \sqrt{3}$ is irrational.

BWOC (By way of contradiction) suppose

$$\sqrt{5} + \sqrt{3} = x \in \mathbb{Q}$$

Squaring

$$5 + 2\sqrt{15} + 3 = x^2$$
$$2\sqrt{15} = x^2 - 8$$

Square again

$$60 = x^4 - 16x^2 + 64$$
$$0 = x^4 - 16x^2 + 4 = f(x)$$

RRT $\Rightarrow$ only possible roots are:

$$\pm 4, \pm 1, \pm 2$$

Checking shows none work. $\blacksquare$.

(Eg: $f(\pm 1) = -11 \neq 0$).

Conjugate Roots Theorem (CJRT)

If $c \in \mathbb{C}$ is a root of a polynomial $p(x) \in \mathbb{R}[x]$ (over $\mathbb{C}$) then $\bar{c}$ is a root of $p(x)$

Pf: Write $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ & $p(c) = 0$. Then

$$
\begin{aligned}
p(\bar{c}) &= a_n(\bar{c})^n + \cdots + a_1 \bar{c} + a_0 \\
&= \overline{a_n c^n} + \cdots + \overline{a_1 c} + \overline{a_0} \quad \text{PM} \\
&= \overline{a_n c^n + \cdots + a_1 c + a_0} \quad \text{PM} \\
&= \overline{p(c)} \\
&= 0.
\end{aligned}
$$