L41P1

Compute the quotient and the remainder when

$$x^4 + 2x^3 + 2x^2 + 2x + 1$$

is divided by $g(x) = 2x^2 + 3x + 4$ in $\mathbb{Z}_5[x]$.

$3x^2 + 4x + 4$ quotient.

$2x^2+3x+4 \,\overline{\smash{)}\, x^4+2x^3+2x^2+2x+1}$

$-(x^4+4x^3+2x^2)$

$3x^3+0x^2+2x$

$-(3x^3+2x^2+x)$

$3x^2+x+1$

$-(3x^2+2x+1)$

$4x$

remainder

Proposition

Let $f(x), g(x) \in \mathbb{F}[x]$. If $f(x) | g(x)$ & $g(x) | f(x)$ then $f(x) = c \cdot g(x)$ for some $c \in \mathbb{F}$.

Pf: Note $f(x)=0$ iff $g(x)=0$. In this case, choose $c=1$. Now, assume neither are 0. By def'n $\exists$ $q(x), \hat{q}(x) \in \mathbb{F}[x]$ s.t.

$$(1) \quad f(x) = g(x) q(x)$$

$$(2) \quad g(x) = f(x) \hat{q}(x)$$

Substitute (2) into (1) giving:

$$f(x) = f(x) \hat{q}(x) q(x)$$

$$f(x) \left(1 - \hat{q}(x) q(x)\right) = 0$$

As $f(x) \neq 0$, we see that

$$1 = \hat{q}(x) q(x)$$

In fact, $\hat{q}(x)$ $q(x)$ are nonzero.

Now, $\deg(1) = 0$ & thus

$$0 = \deg(\hat{q}(x)q(x)) = \deg(\hat{q}(x)) + \deg(q(x))$$

(Exercise)

$\therefore \deg(q(x)) = 0 = \deg(\hat{q}(x))$

$\therefore q(x) = c \in \mathbb{F}$. Thus, by (1)

$$f(x) = c\,g(x).$$

Remainder Theorem (RT)

Suppose $f(x) \in \mathbb{F}[x]$ and $c \in \mathbb{F}$. Then the remainder when $f(x)$ is divided by $x - c$ is $f(c)$.

Pf: By DAP, $\exists!\, q(x), r(x) \in \mathbb{F}[x]$ s.t

$$f(x) = (x-c)q(x) + r(x) \qquad (3)$$

with $r(x) = 0$ OR $\deg(r(x)) < \underbrace{\deg(x-c)}_{=1}$

$$\therefore \deg(r(x)) = 0$$

Hence, in either case, $r(x)=K$ for some $K \in \mathbb{F}$. Plug $x=c$ into (3) to see that $f(c)=r(c)=K$

Hence $r(x)=f(c)$. ∎

Ex: Find the remainder when $f(z)=z^2+1$ is divided by

A) $z-1$ B) $z+1=z-(-1)$ C) $z+i+1=z-(-i-1)$

Sol'n: A) By RT, remainder is $f(1)=(1)^2+1=2$

B) By RT, remainder is $f(-1)=(-1)^2+1=2$

Note: $z^2+1=(z-1)(z+1)+2$.

C) By RT, remainder is

$$f(-i-1)=(-i-1)^2+1=-1+2i+1+1$$
$$=2i+1.$$

In $\mathbb{Z}_7[x]$, what is the remainder when $4x^3 + 2x + 5$ is divided by $x + 6$?

Sol'n: $x+6 = x-1$. By RT, the remainder is

$$4(1)^3+2(1)+5 = 11$$
$$\equiv 4 \mod 7.$$

Factor Theorem (FT)

Suppose $f(x) \in \mathbb{F}[x]$ & $c \in \mathbb{F}$.
The polynomial $x-c$ is a factor of $f(x)$ iff $f(c)=0$ ie c is a root of $f(x)$

Pf: $x-c$ is a factor of $f(x)$

$\iff r(x)=0$

$\iff f(c)=0$ by RT. $\blacksquare$

Prove that there does not exist a real linear factor of

$$f(x) = x^8 + x^3 + 1.$$

Pf: By FT, it suffices to show $f(x)$ has <u>no</u> real roots. We will show $f(x) > 0$ $\forall x \in \mathbb{R}$.

If $|x| \geq 1$, then $x^8 + x^3 \geq 0$

hence $f(x) > 0$

If $|x| < 1$, then $|x^3| < 1$

hence $x^3 + 1 > 0$

hence $f(x) > 0$.