Lecture 39

Theorem: Complex nth Roots Theorem (CNRT) Any nonzero complex number has exactly $n \in \mathbb{N}$ distinct nth roots. The roots lie on a circle of radius $|z|$ centred at the origin and spaced out evenly by angles of $2\pi/n$. Concretely, if $a = re^{i\theta}$, then solutions to $z^n = a$ are given by $z = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}$ for $k \in \{0, 1, ..., n-1\}$.

Proof: The proof is like the example yesterday and is left as additional reading. ■

Definition: An nth root of unity is a complex number z such that $z^n = 1$. These are sometimes denoted by ζ_n.

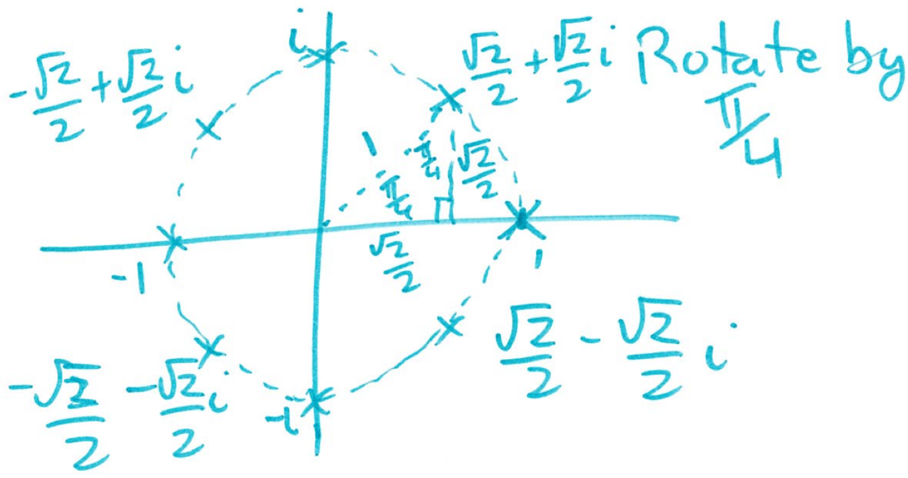
Example: -1 is a second root of unity (and a fourth root of unity and a sixth root of unity etc.)

Instructor's Comments: This is the 10 minute mark; though likely the previous lecture spilled over to this lecture.

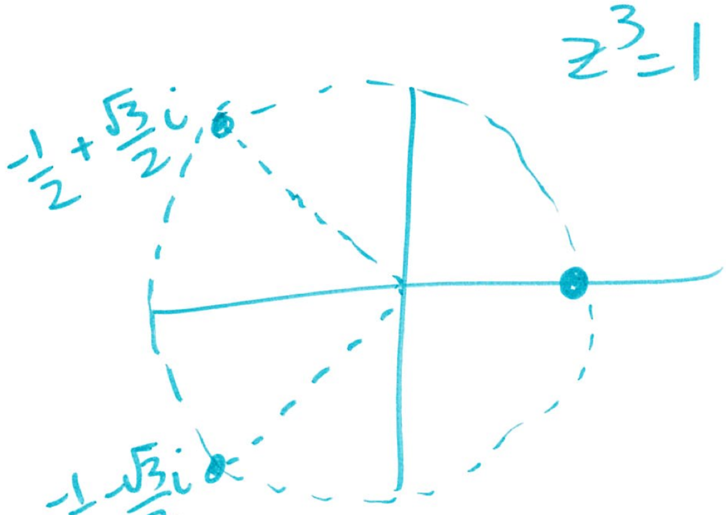
Handout or Document Camera or Class Exercise

Find all eighth roots of unity in standard form.

Solution: We want to solve $z^8 = 1$. We know that $\{\pm 1, \pm i\}$ are solutions. We can draw to find the rest:



For another example, look at $z^3 = 1$:



Example: Solve $z^5 = -16\overline{z}$.

Instructor's Comments: Get students to guess the total number of solutions. Also get them to find a solution by inspection. The answer is surprising!

Solution: This is a tricky problem. One could convert to polar coordinates but I prefer to reason as follows. If I can't solve the equation as written, maybe I can simplify by taking lengths on both sides.

$$|z^5| = |z|^5 = |-16\overline{z}| = 16|\overline{z}| = 16|z|$$

This gives $|z|^5 = 16|z|$. Hence $|z|^5 - 16|z| = 0$ giving $|z|(|z|^4 - 16) = 0$. This gives either $|z| = 0$ which translates to $z = 0$ or $|z|^4 = 16$ which gives $|z| = 2$. So assuming that $z \neq 0$, we multiply the original equation by z to yield

$$z^6 = -16z\overline{z} = -16|z|^2 = -64$$

but this question we solved before! Therefore,

$$z \in \{0, \pm 2i, \pm\sqrt{3} \pm i\}$$

Thus, there are seven solutions!

Instructor's Comments: This is the 40 minute mark; if you spilled over from the previous lecture, this is the 50 minute mark. Otherwise do the next problem (which is one we did before)

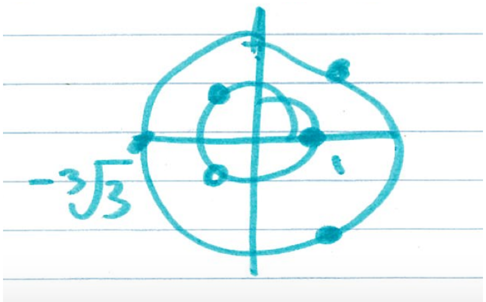
Example: Solve $z^6 + 2z^3 - 3 = 0$.

Proof: From before, we factored this to $(z^3 - 1)(z^3 + 3) = 0$ and thus $z^3 = 1$ or $z^3 = -3$. From CNRT, we see that the solutions to $z^3 = 1 = \cos(0) + i\sin(0)$ are given by

$$z \in \{e^{i\cdot 0}, e^{i\cdot 2\pi/3}, e^{i\cdot 4\pi/3}\}$$

and solutions to $z^3 = -3 = 3(\cos(\pi) + i\sin(\pi))$ are given by

$$z \in \{\sqrt[3]{3}e^{i\cdot\pi/3}, \sqrt[3]{3}e^{i\cdot\pi}, \sqrt[3]{3}e^{i\cdot 5\pi/3}\}$$



This completes the question. ■