Lecture 38

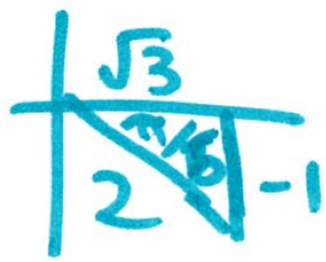
Handout or Document Camera or Class Exercise

Write $(\sqrt{3}-i)^{10}$ in standard form.

Solution: Convert $\sqrt{3}-i$ to polar coordinates.

$$\begin{aligned}\sqrt{3}-i &= 2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\\ &= 2\text{cis}(-\pi/6)\\ &= 2\text{cis}(11\pi/6)\end{aligned}$$

seen via the diagram



Lastly,

$$\begin{aligned}(2\text{cis}(11\pi/6))^{10} &= 2^{10}\text{cis}(110\pi/6) && \text{DMT}\\ &= 2^{10}\text{cis}(55\pi/3)\\ &= 2^{10}\text{cis}(9(2\pi)+\pi/3)\\ &= 2^{10}\text{cis}(\pi/3)\\ &= 2^{10}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= 2^9+2^9\sqrt{3}i\\ &= 512+512\sqrt{3}i\end{aligned}$$

seen via the diagram

Instructor's Comments: This is the 10-15 minute mark

Complex Exponential Function

Definition: For a real θ, define

$$e^{i\theta} := \cos(\theta) + i\sin(\theta) = \text{cis}(\theta)$$

Note: Can write $z \in \mathbb{C}$ as $z = re^{i\theta}$ where $r = |z|$ and θ is an argument of z.

Question: Why is this definition reasonable? While we can't prove the answer to this question, we can give convincing arguments.

Reason 1: Exponential Laws Work! For $\theta, \alpha \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta+\alpha)} \qquad \text{PMCN}$$
$$(e^{i\theta})^n = e^{in\theta} \qquad \text{DMT}$$

Reason 2: Derivative with respect to θ makes sense.

$$\begin{aligned}\frac{d}{d\theta}(\cos(\theta) + i\sin(\theta)) &= -\sin(\theta) + i\cos(\theta)\\ &= i(\cos(\theta) + i\sin(\theta))\\ &= ie^{i\theta}\end{aligned}$$

Reason 3: Power series.

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\end{aligned}$$

Using these and combining gives

$$e^{ix} = \cos(x) + i\sin(x)$$

Setting $\theta = \pi$ gives Euler's Formula:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

Instructor's Comments: This is the 25 minute mark.

Example: Write $(2e^{11\pi/6})^6$ in standard form.

Solution: By exponent rules (DMT), we have

$$\begin{aligned}(2e^{11\pi/6})^6 &= 2^6 e^{11\pi i}\\ &= 2^6(\cos(11\pi) + i\sin(11\pi))\\ &= 2^6(-1 + 0i)\\ &= -64\end{aligned}$$

Instructor's Comments: This is the 30 minute mark.

Example: Solve $z^6 + 2z^3 - 3 = 0$

Solution: Factoring gives

$$0 = z^6 + 2z^3 - 3 = (z^3 - 1)(z^3 + 3)$$

Hence $z^3 = 1$ or $z^3 = -3$.

Question: Can we solve $z^n = w$ for a fixed $w \in \mathbb{C}$?

Note: We saw a case of this already with $n = 2$ and $w = -r$. We'll delay the previous question until later.

Example: Solve $z^6 = -64$.

Solution: We already saw that $2e^{11\pi/6}$ was a solution. Note that $\pm 2i$ are two others found by inspection. How do we find all the solutions in general? The answer involves using polar coordinates. Write $z = re^{i\theta}$. Then

$$z^6 = r^6 e^{i6\theta} = -64$$

Taking the modulus yields $|r|^6 |e^{i6\theta}| = 64$. Since for any real α, we have

$$|e^{i\alpha}| = |\cos(\alpha) + i\sin(\alpha)| = \sqrt{\cos^2(\alpha) + \sin^2(\alpha)} = 1$$

we see that $|r|^6 = 64$ and hence $r = 2$ since r is a positive real number.

Instructor's Comments: This is the 40 minute mark.

Hence, we see that $-64 = r^6 e^{i6\theta} = 64e^{i6\theta}$ and so $e^{i6\theta} = -1$. Thus,

$$\cos(6\theta) + i\sin(6\theta) = -1 = \cos(\pi) + i\sin(\pi)$$

Hence, this is true when $6\theta = \pi + 2\pi k$ for all $k \in \mathbb{Z}$. Solving for θ gives

$$\theta = \frac{\pi + 2\pi k}{6} = \frac{\pi}{6} + \frac{\pi}{3}k$$

Now, when do two values of θ coincide with the same complex point? Answer: When they differ by multiples of 2π.

Claim: $\theta_1 = \frac{\pi}{6} + \frac{\pi}{3}k_1$ and $\theta_2 = \frac{\pi}{6} + \frac{\pi}{3}k_2$ are equal up to 2π rotations if and only if $k_1 \equiv k_2 \pmod 6$.

Proof: We have that

$$\begin{aligned} \theta_1 &= \theta_2 + 2\pi m \qquad \text{for some } m \in \mathbb{Z} \\ \frac{\pi}{6} + \frac{\pi}{3}k_1 &= \frac{\pi}{6} + \frac{\pi}{3}k_2 + 2\pi m \\ \frac{\pi}{3}k_1 &= \frac{\pi}{3}k_2 + 2\pi m \\ k_1 &= k_2 + 6m \\ k_1 &\equiv k_2 \pmod 6 \end{aligned}$$

and each of the above steps are if and only if steps. This completes the proof of the claim.

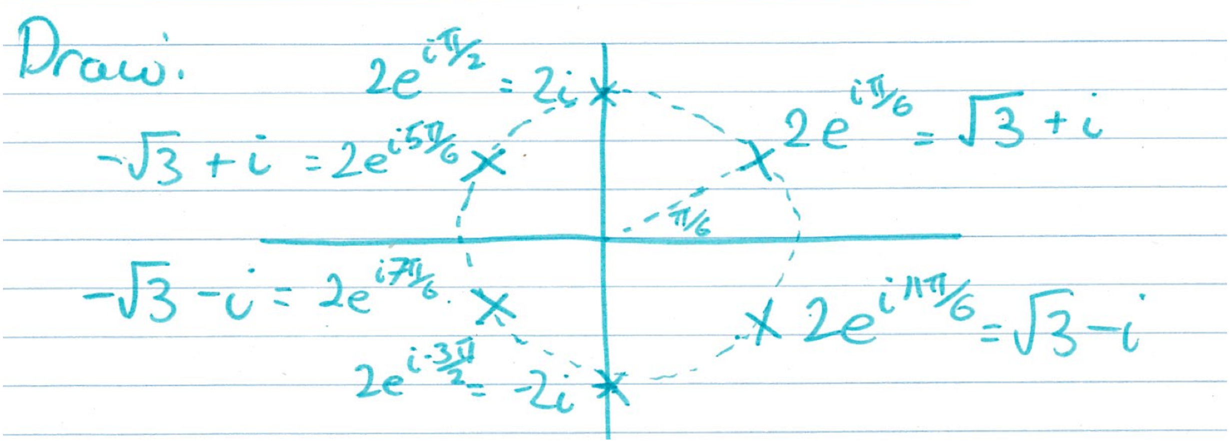
Hence $\theta = \frac{\pi}{6} + \frac{\pi}{3}k_1$ for $k_1 \in \{0, 1, 2, 3, 4, 5\}$. Thus,

$$\theta \in \{\tfrac{\pi}{6}, \tfrac{3\pi}{6}, \tfrac{5\pi}{6}, \tfrac{7\pi}{6}, \tfrac{9\pi}{6}, \tfrac{11\pi}{6}\}$$

or rewritten as

$$\theta \in \{\tfrac{\pi}{6} + \tfrac{\pi}{3}k_1 : k_1 \in \{0, 1, 2, 3, 4, 5\}\}$$

Therefore, $z = re^{i\theta} \in \{2e^{i(\pi/6+\pi k/3)} : k \in \{0, 1, 2, 3, 4, 5\}\}$.



Instructor's Comments: In all likelihood, the 50 minute mark is somewhere above. Carry through to lecture 39 as needed.