De Moivre's Theorem

Let $\theta \in \mathbb{R}$ & $n \in \mathbb{Z}$. Then

$$(\text{cis}\,\theta)^n = \text{cis}(n\theta)$$

Pf: From work yesterday, it suffices to prove the claim for $n \in \mathbb{N}$.

Proof by induction on n.

Base Case: $n=1$

$\text{cis}(n\theta) = \text{cis}\,\theta = (\text{cis}\,\theta)^1 = (\text{cis}\,\theta)^n$.

IH: Assume that

$$(\text{cis}\,\theta)^k = \text{cis}(k\theta)$$

for some $k \in \mathbb{N}$

I Step: WANT $(\text{cis}\,\theta)^{k+1} = \text{cis}((k+1)\theta)$

$$\begin{aligned} \text{LHS} = (\text{cis}\,\theta)^{k+1} &= (\text{cis}\,\theta)^k(\text{cis}\,\theta) \\ &\overset{\text{IH}}{=} \text{cis}(k\theta)\,\text{cis}(\theta) \\ &\overset{\text{PMCN}}{=} \text{cis}(k\theta+\theta) \\ &= \text{cis}((k+1)\theta) \end{aligned}$$

$\therefore$ by POMI $(\text{cis}\,\theta)^n = \text{cis}(n\theta)\ \forall n \in \mathbb{N}$. $\blacksquare$

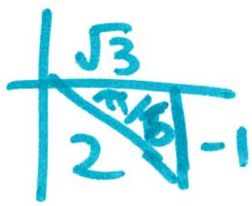
Corollary: If $z = r\,\text{cis}\,\theta$ then

$$z^n = r^n\,\text{cis}(n\theta)$$

Write $(\sqrt{3} - i)^{10}$ in standard form.

Convert $\sqrt{3} - i$ to polar coordinates.

$$\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \quad \because 2 = |\sqrt{3} - i|$$



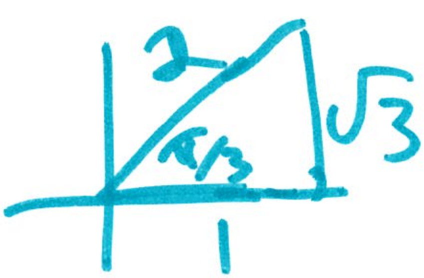
$$= 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

$$= 2\operatorname{cis}\left(\frac{11\pi}{6}\right)$$

$$\left(2\operatorname{cis}\left(\frac{11\pi}{6}\right)\right)^{10} \overset{\text{DMT}}{=} 2^{10}\operatorname{cis}\left(\frac{110}{6}\pi\right)$$

$$= 2^{10}\operatorname{cis}\left(\frac{55}{3}\pi\right)$$

$$= 2^{10}\operatorname{cis}\left(9(2\pi) + \frac{\pi}{3}\right)$$



$$= 2^{10}\operatorname{cis}\left(\frac{\pi}{3}\right)$$

$$= 2^{10}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$= 2^{9} + 2^{9}\sqrt{3}\, i$$

$$= 512 + 512\sqrt{3}\, i$$

Complex Exponential Function

For real θ, define

$$e^{i\theta} := \cos\theta + i\sin\theta = \mathrm{cis}\,\theta$$

Note: Can write $z \in \mathbb{C}$ as $z = re^{i\theta}$ where $r = |z|$ & θ is an argument of z

Q: Why this def'n?

Reason 1: Exponent Laws Work!

$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta+\alpha)} \quad \text{(PMCN)}$$

$n \in \mathbb{Z}$ $$(e^{i\theta})^n = e^{in\theta} \quad \text{(DMT)}$$

Reason 2: Derivative wrt θ

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$$
$$= i(\cos\theta + i\sin\theta)$$
$$= ie^{i\theta}$$

Reason 3: Power Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Using these,

$$e^{ix} = \cos x + i \sin x$$ Euler's Formula.

If $\theta = \pi$ then

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

Ex: Write $(2e^{11\pi i/6})^6$ in standard form.

Sol'n: By exponent rules (DMT)

$$\begin{aligned}(2e^{11\pi i/6})^6 &= 2^6 e^{11\pi i} \\ &= 2^6(\cos(11\pi) + i\sin(11\pi)) \\ &= 2^6(-1 + 0 \cdot i) \\ &= -64.\end{aligned}$$

Solve: $z^6+2z^3-3=0$

$$(z^3)^2+2z^3-3=0$$

$$(z^3-1)(z^3+3)=0$$

$z^3=1$ OR $z^3=-3$

Q: Can we solve $z^n=w$ for a fixed $w\in\mathbb{C}$?

Note: Saw this with $n=2$ & $w=-r$.

Ex: Solve $z^6=-64$

Sol'n: $2e^{11\pi i/6}$ was a solution.

$\pm 2i$ are 2 other examples.

How do we find solutions in general?

Ans: Write $z=re^{i\theta}$

$$z^6=r^6e^{i6\theta}=-64$$

$$|r|^6|e^{i6\theta}|=64$$

$$|r|^6=64$$

$$\Rightarrow r=2 \quad (\because r>0)$$