Lecture 37

Handout or Document Camera or Class Exercise

Express the following in terms of polar coordinates:

(i) -3

(ii) $1 - i$

Solution:

(i) Note that $r = |-3| = 3$ and $\theta = \arctan(0/-3) = 0$. Then, since -3 lives between the second and third quadrant, you need to add π to the previous answer. Thus $\theta = \pi$ and hence $-3 = 3\text{cis}(\pi)$.

Instructor's Comments: Make sure to note the addition of pi above.

(ii) Note that $r = |1 - i| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Hence

$$\begin{aligned} 1 - i &= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \\ &= \sqrt{2}(\cos(7\pi/4) + i\sin(7\pi/4)) \\ &= \sqrt{2}\text{cis}(7\pi/4) \end{aligned}$$

Instructor's Comments: This is the 10 minute mark.

Handout or Document Camera or Class Exercise

(i) Write $\text{cis}(15\pi/6)$ in standard form.

(ii) Write $-3\sqrt{2} + 3\sqrt{6}i$ in polar form.

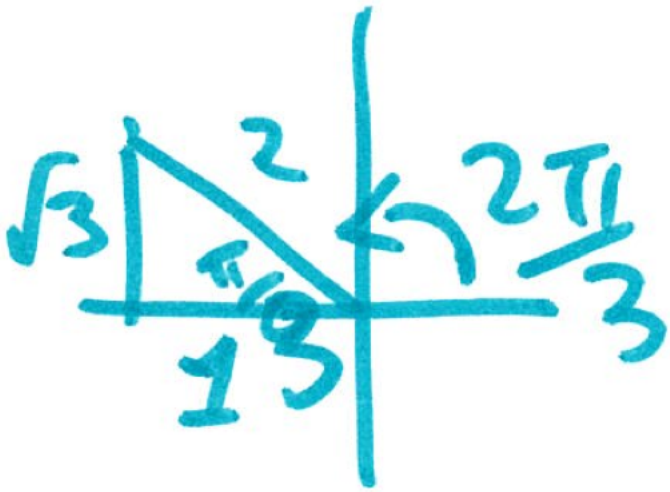
Solution:

(i) $\text{cis}(15\pi/6) = \cos(5\pi/2) + i\sin(5\pi/2) = i.$

(ii) Note that

$$\begin{aligned} r &= |-3\sqrt{2} + 3\sqrt{6}i| \\ &= \sqrt{(-3\sqrt{2})^2 + (3\sqrt{6})^2} \\ &= \sqrt{18 + 54} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

Therefore, $-3\sqrt{2}+3\sqrt{6}i = 6\sqrt{2}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = 6\sqrt{2}\text{cis}(2\pi/3)$ where the last equality holds since



Instructor's Comments: This is the 20 minute mark

Theorem: (Polar Multiplication of Complex Numbers (PMCN)) If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then

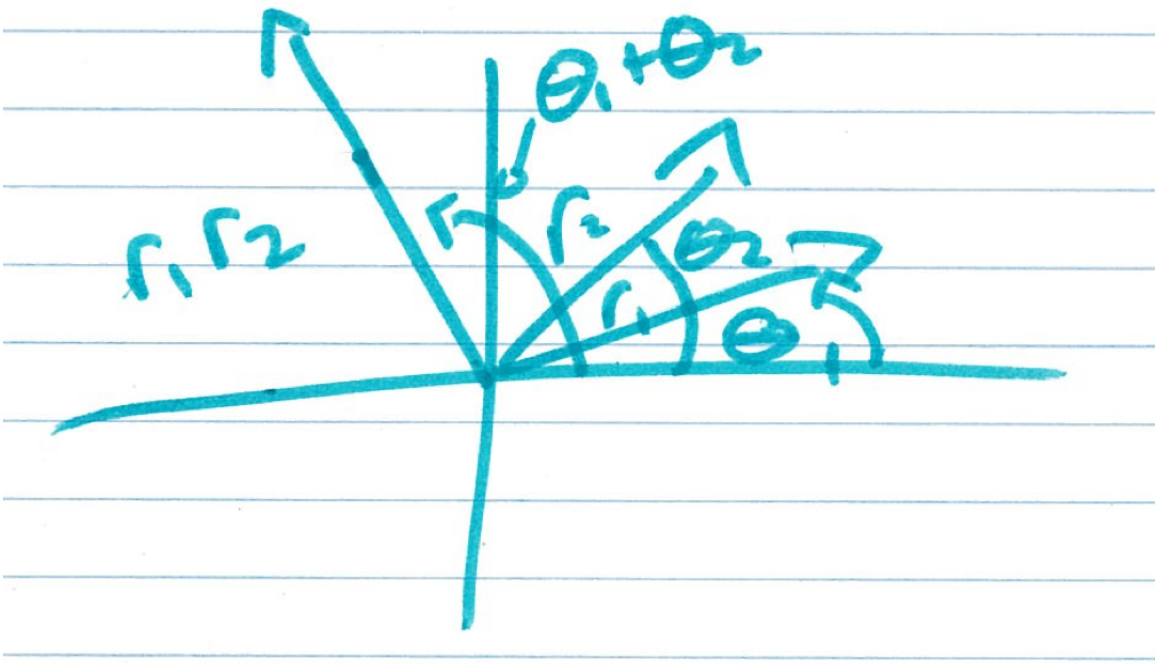
$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$$

Proof: We have

$$\begin{aligned} z_1 z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 \text{cis}(\theta_1 + \theta_2) \end{aligned}$$

where in line 3 above, we used trig identities. This completes the proof. ∎

Corollary: Multiplication by $i = \cos(\pi/2) + i\sin(\pi/2)$ gives a rotation by $\pi/2$.



Example: Using Polar Multiplication of Complex Numbers on $(\sqrt{6}+\sqrt{2}i)(-3\sqrt{2}+3\sqrt{6}i)$ gives

$$\begin{aligned} (\sqrt{6} + \sqrt{2}i)(-3\sqrt{2} + 3\sqrt{6}i) &= 2\sqrt{2}\text{cis}(\pi/6) \cdot 6\sqrt{2}\text{cis}(2\pi/3) \\ &= 24\text{cis}(\pi/6 + 2\pi/3) && \text{By PMCN} \\ &= 24\text{cis}(5\pi/6) \\ &= 24(-\sqrt{3}/2 + i/2) \\ &= -12\sqrt{3} + 12i \end{aligned}$$

Instructor's Comments: This is the 30-35 minute mark.

Theorem: (De Moivre's Theorem (DMT)) If $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, then

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

or more compactly,

$$\text{cis}(\theta)^n = \text{cis}(n\theta).$$

Instructor's Comments: Emphasize here that we want to use induction but need to reduce to the natural numbers first

Proof: First note that when $n = 0$, we see that $(\cos(\theta) + i\sin(\theta))^0 = 1$ and that $\cos(0\theta) + i\sin(0\theta) = 1$ so the statement holds. For $n > 0$, we proceed by induction on n. For the base case, consider $n = 1$. Then

$$(\cos(\theta) + i\sin(\theta))^n = \cos(\theta) + i\sin(\theta) = \cos(n\theta) + i\sin(n\theta).$$

Now, assume that

$$(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$$

holds for some $k \in \mathbb{N}$. For the inductive step, note that

$$\begin{aligned}
(\cos(\theta) + i\sin(\theta))^{k+1} &= (\cos(\theta) + i\sin(\theta))^k(\cos(\theta) + i\sin(\theta)) \\
&= (\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta)) && \text{Inductive hypothesis} \\
&= \cos((k+1)\theta) + i\sin((k+1)\theta) && \text{By PMCN}
\end{aligned}$$

For $n < 0$, we write $n = -m$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
\text{cis}(\theta)^n &= \text{cis}(\theta)^{-m} \\
&= (\text{cis}(\theta)^m)^{-1} \\
&= \text{cis}(m\theta)^{-1} \\
&= \frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)} && \text{Since } z^{-1} = \overline{z}/|z|^2 \\
&= \cos(m\theta) - i\sin(m\theta)
\end{aligned}$$

and $\cos(-m\theta) + i\sin(-m\theta) = \cos(m\theta) - i\sin(m\theta)$ since cosine is even and sine is odd. This completes the proof. ∎

Corollary: If $z = r\text{cis}(\theta)$ then $z^n = r^n\text{cis}(n\theta)$.

Instructor's Comments: This is the 50 minute mark.