Lecture 36

Handout or Document Camera or Class Exercise

Instructor's Comments: There are two clicker questions here. Choose the one you prefer. I like the first one because students often forget they can use LDEs to find inverses.

Let $[x]$ be the inverse of $[241]$ in $\mathbb{Z}_{1001}$, if it exists, where $0 \le x < 1001$. Determine the sum of the digits of x.

A) 7

B) 9

C) 11

D) 12

E) $[x]$ does not exist

Solution: We use the Extended Euclidean Algorithm (EEA) on $241x + 1001y = 1$ to see that

x	y	r	q
0	1	1001	0
1	0	241	0
-4	1	37	$\lfloor \frac{1001}{241} \rfloor = 4$
25	-6	19	$\lfloor \frac{241}{37} \rfloor = 6$
-29	7	18	$\lfloor \frac{37}{19} \rfloor = 1$
54	-13	1	$\lfloor \frac{19}{18} \rfloor = 1$

Hence $241(54) + 1001(-13) = 1$ and so $[54]$ is the inverse of $[241]$ in $\mathbb{Z}_{1001}$. Since $5 + 4 = 9$, the correct answer is B.

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How many integers x satisfy all of the following three conditions?

$$x \equiv 6 \pmod{13}$$
$$4x \equiv 3 \pmod{7}$$
$$-1000 < x < 1000$$

A) 1

B) 7

C) 13

D) 22

E) 91

Solution: Note that multiplying $4x \equiv 3 \pmod 7$ by 2 gives $x \equiv 6 \pmod 7$. By the Chinese Remainder Theorem or by Splitting the Modulus, we see that $x \equiv 6 \pmod{91}$. Thus, $x = 6 + 91k$. Using this with the range restriction gives

$$-1000 < 6 + 91k < 1000$$
$$-1006 < 91k < 994$$

Note that $91 \cdot 10 = 910$ and $91 \cdot 11 = 1001$. Therefore, the above condition with the fact that $k \in \mathbb{Z}$ reduces to $-11 \leq k \leq 10$ and thus, there are 22 solutions.

Instructor's Comments: This is the 10 minute mark; this is a longer problem

Definition: The modulus of $z = x + yi$ is the nonnegative real number

$$|z| = |x + yi| := \sqrt{x^2 + y^2}$$

Proposition: (Properties of Modulus (PM))

(i) $|\overline{z}| = |z|$

(ii) $z\overline{z} = |z|^2$

(iii) $|z| = 0 \Leftrightarrow z = 0$

(iv) $|zw| = |z||w|$

(v) $|z + w| \leq |z| + |w|$ (This is called the triangle inequality)

Instructor's Comments: Mention that properties 3,4,5 define a norm. I recommend not doing the proof of all of these. I would do 2,4 and 5. In fact, I would make 5 an in-class reading proof to get some reading practice in.

Proof: Throughout, let $z = x + yi$.

(i) Note that

$$|\overline{z}| = |x - yi| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

(ii) $z\overline{z} = (x + yi)(x - yi) = x^2 + y^2 = |z|^2$

(iii) $|z| = 0$ if and only if $\sqrt{x^2 + y^2} = 0$ if and only if $x^2 + y^2 = 0$ if and only if $x = y = 0$ if and only if $z = 0$.

(iv) Using the second property above and Properties of Conjugates, we have

$$|zw|^2 = (zw)\overline{zw} = z\overline{z}w\overline{w} = |z|^2|w|^2$$

Hence, since all the numbers above are real, we have that $|zw| = |z||w|$.

(v) (See the handout on next page)

Handout or Document Camera or Class Exercise

To prove $|z+w| \leq |z| + |w|$, it suffices to prove that

$$|z+w|^2 \leq (|z|+|w|)^2 = |z|^2 + 2|zw| + |w|^2$$

since the modulus is a positive real number. Using the Properties of Modulus and the Properties of Conjugates, we have

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) && \text{PM} \\
&= (z+w)(\bar{z}+\bar{w}) && \text{PCJ} \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 && \text{PCJ and PM}
\end{aligned}$$

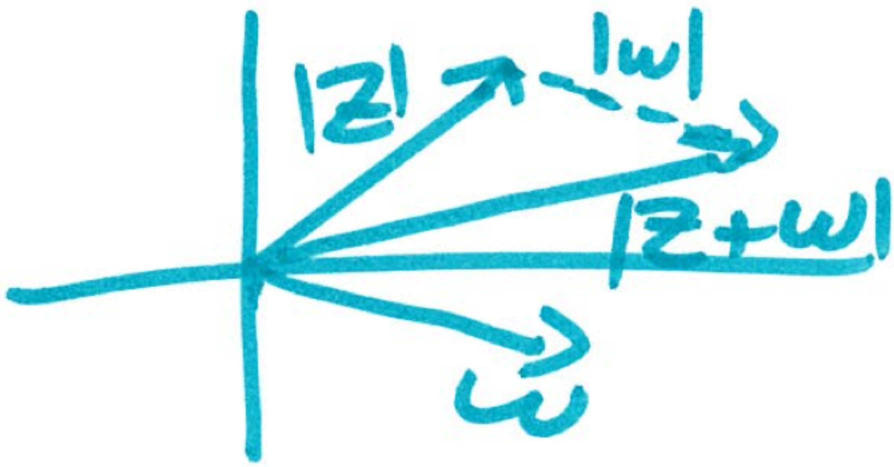
Now, from Properties of Conjugates, we have that

$$z\bar{w} + \overline{z\bar{w}} = 2\Re(z\bar{w}) \leq 2|z\bar{w}| = 2|zw|$$

and hence

$$|z+w|^2 = |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \leq |z|^2 + 2|zw| + |w|^2$$

completing the proof.



Instructor's Comments: This is the 25-30 minute mark

Revisit Inverses

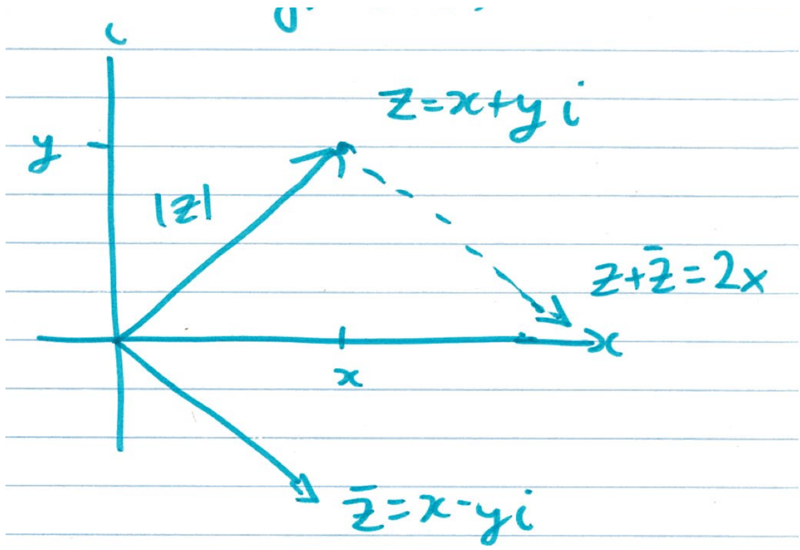
Recall, we defined the inverse of z by

$$z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$$

Note that

$$z^{-1} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2}$$

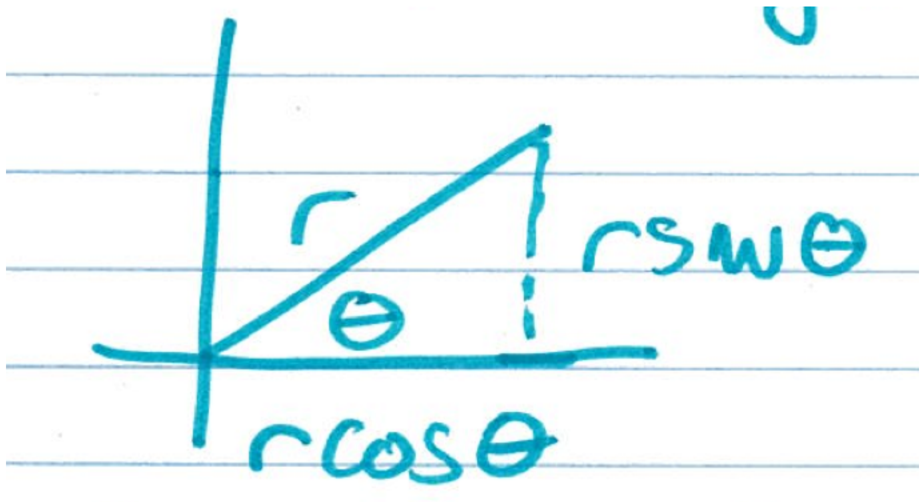
Argand Diagram



Instructor's Comments: This is the 35 minute mark.

Polar Coordinates

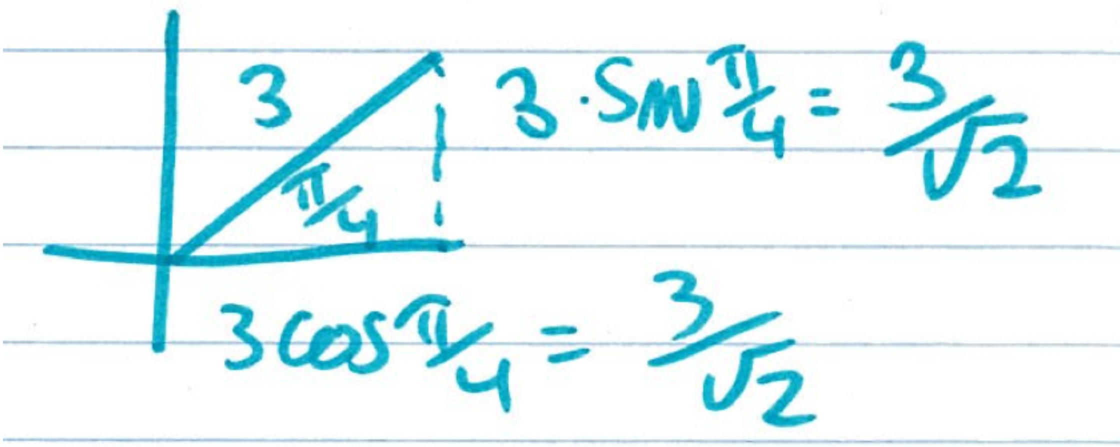
A point in the plane corresponds to a length and an angle:



Example: $(r, \theta) = (3, \frac{\pi}{4})$ corresponds to

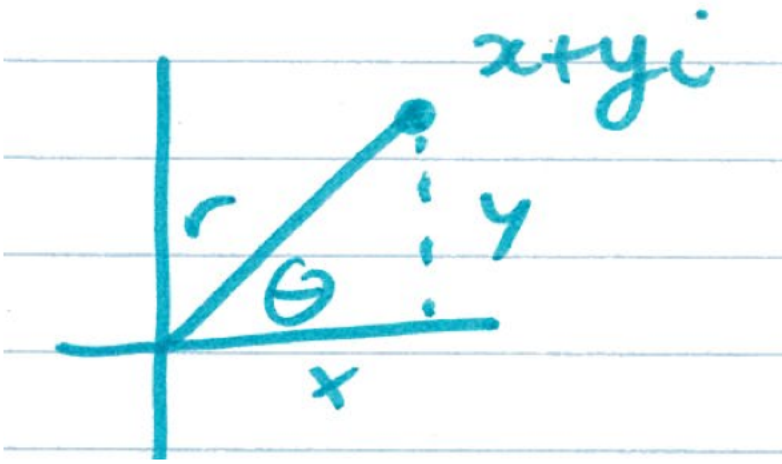
$$3\cos(\pi/4) + i(3\sin(\pi/4)) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$$

via the picture



Given $z = x + yi$, we see that

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\theta = \arccos(x/r) = \arcsin(y/r) = \arctan(y/x)$$



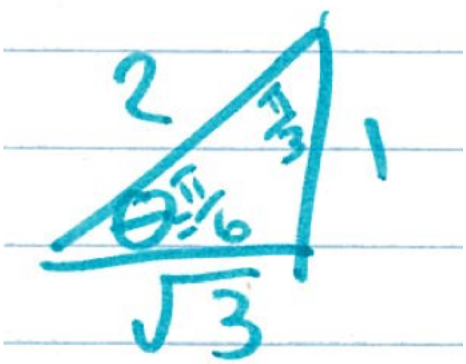
Note: WARNING. The angle θ might be $\arctan(y/x)$ OR $\pi + \arctan(y/x)$ depending on which quadrant we are in. More on this next class.

Example: Write $z = \sqrt{6} + \sqrt{2}i$ using polar coordinates.

Solution: Note that $r = \sqrt{\sqrt{6}^2 + \sqrt{2}^2} = \sqrt{8} = 2\sqrt{2}$. Further,

$$\arctan(\sqrt{2}/\sqrt{6}) = \arctan 1/\sqrt{3} = \pi/6$$

Note: There is no need to add π to the above answer since the answer lies in the first quadrant.



Therefore, z corresponds to $(r, \theta) = (2\sqrt{2}, \pi/6)$. ■

Definition: The polar form of a complex number z is $z = r(\cos(\theta) + i\sin(\theta))$ where r is the modulus of z and θ is called *an* argument of z. This is sometimes denoted by $\arg(z) = \theta$. Further, denote $\text{cis}(\theta) := \cos(\theta) + i\sin(\theta)$.

Example: If $z = \sqrt{6} + \sqrt{2}i$, then $z = 2\sqrt{2}(\cos(\pi/6) + i\sin(\pi/6)) = 2\sqrt{2}\text{cis}(\pi/6)$.

Instructor's Comments: This is the 50 minute mark.