Properties of Conjugates (PCJ)

Let $z, w \in \mathbb{C}$.

1. $\overline{z+w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\,\bar{w}$
3. $\bar{\bar{z}} = z$
4. $z + \bar{z} = 2\mathrm{Re}(z)$
5. $z - \bar{z} = 2i\mathrm{Im}(z)$

Prove the following for $z \in \mathbb{C}$

1. $z \in \mathbb{R}$ if and only if $z = \bar{z}$.

2. z is purely imaginary if and only if $z = -\bar{z}$.

1. $\Rightarrow$ Let $z = x + 0i \in \mathbb{R}$.

Then $\bar{z} = x - 0i = x = z$

$\Leftarrow$ Let $z = x + yi$, $x, y \in \mathbb{R}$. Assume

$$z = \bar{z}$$

$$x + yi = x - yi$$

$$\Rightarrow \quad y = -y$$

$$\Rightarrow \quad 2y = 0$$

$$\Rightarrow \quad y = 0 \qquad \therefore z = x + 0i \in \mathbb{R}.$$

2. z is purely imaginary

$\Leftrightarrow \; iz \in \mathbb{R}$

By 1. $\Leftrightarrow \; iz = \overline{iz}$

$\Leftrightarrow \; iz = -i\bar{z}$

$\Leftrightarrow \; z = -\bar{z}$

Def'n: The modulus of $z = x + yi$ is the non-negative real number

$$|z| = |x+yi| := \sqrt{x^2+y^2}$$

Properties of Modulus (PM)

1. $|\bar{z}| = |z|$
2. $z\bar{z} = |z|^2$
3. $|z| = 0 \iff z = 0$
4. $|zw| = |z||w|$
5. $|z+w| \leq |z| + |w|$ Δ inequality.

Pf of 4: Let $z = x+yi$ & $w = u+vi$

Suffices to show $|zw|^2 = |z|^2|w|^2$

$$\begin{aligned} |zw|^2 &= |(x+yi)(u+vi)|^2 \\ &= |(xu-vy)+(xv+uy)i|^2 \\ &= (xu-vy)^2+(xv+uy)^2. \end{aligned}$$

$$|zw|^2 = |(x+yi)(u+vi)|^2$$
$$= |(xu-vy)+(xv+uy)i|^2$$

By def'n of $|z|$. $\rightarrow$
$$= (xu-vy)^2 + (xv+uy)^2$$
$$= x^2u^2 - 2xuvy + v^2y^2 + x^2v^2 + 2xuvy + u^2y^2$$
$$= x^2u^2 + x^2v^2 + v^2y^2 + u^2y^2$$

$$|z|^2|w|^2 = |x+yi|^2|u+vi|^2$$
$$= (x^2+y^2)(u^2+v^2)$$
$$= x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2 = |zw|^2$$

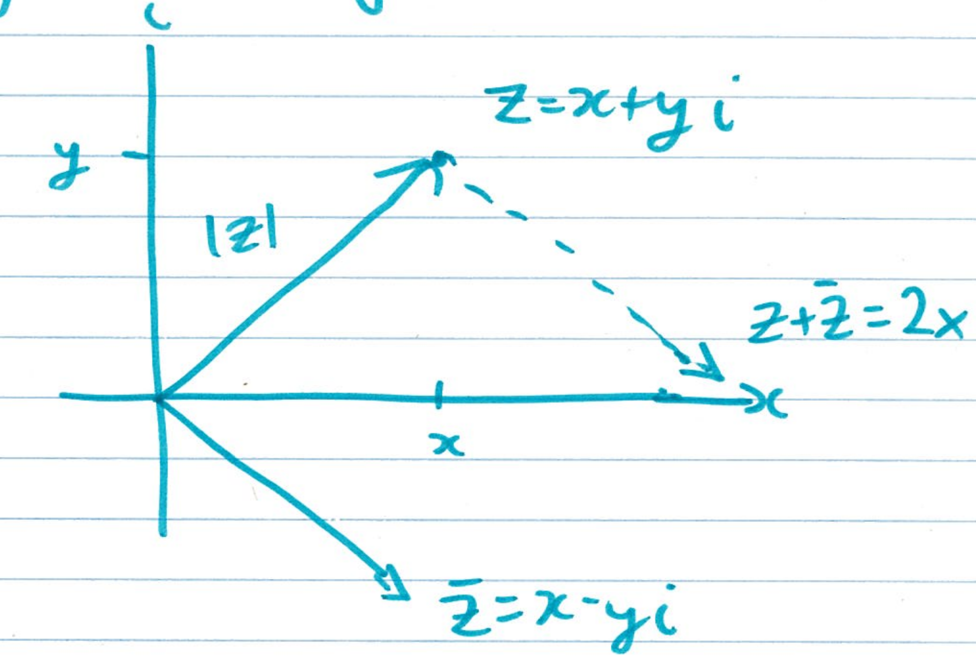
Pf of 5 (Exercise)

Revisit Inverses:

If $z = x+yi$ then $z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$

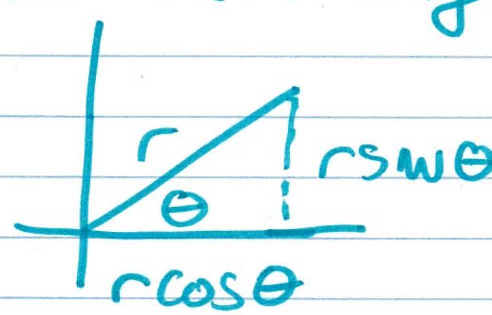
Note $z^{-1} = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\cdot\overline{z}} = \frac{\overline{z}}{|z|^2}$.

Pictures! (Argand Diagrams)

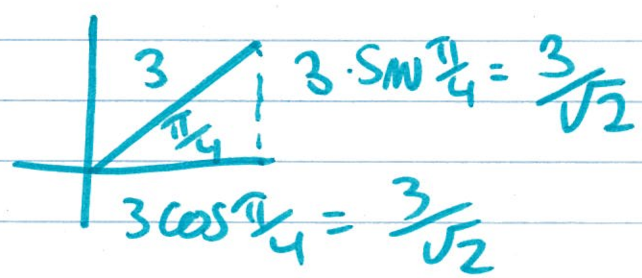


Polar Coordinates

A point in the plane corresponds to a length and an angle.



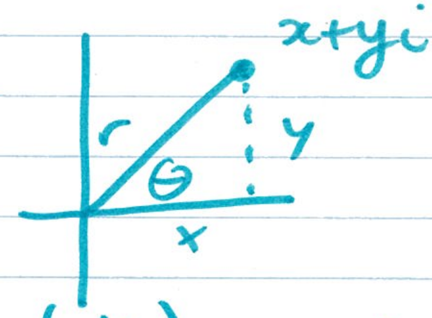
Ex: $(r, \theta) = (3, \frac{\pi}{4})$



Corresponds to $3\cos\frac{\pi}{4} + i \cdot 3\sin\frac{\pi}{4}$

$$= \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$$

Given $z = x + yi$



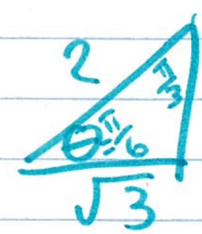
$r = |z| = \sqrt{x^2 + y^2}$

$\theta = \arccos\left(\frac{x}{r}\right) = \arcsin\left(\frac{y}{r}\right) = \arctan\left(\frac{y}{x}\right)$

Ex: $z = \sqrt{6} + \sqrt{2}\, i$

$r = \sqrt{\sqrt{6}^2 + \sqrt{2}^2} = \sqrt{8} = 2\sqrt{2}.$

$\theta = \arctan\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$



$\therefore z$ corresponds to $(r, \theta) = (2\sqrt{2}, \frac{\pi}{6})$

Def'n: The polar form of a complex number z is $z = r(\cos\theta + i\sin\theta)$ where r is the modulus of z and θ is called <u>an</u> argument of z ($\arg(z) = \theta$)

Denote $\operatorname{cis}\theta := \cos\theta + i\sin\theta$

Ex: $z = \sqrt{6} + \sqrt{2}\, i = 2\sqrt{2}\,(\cos \pi/6 + i \sin \pi/6)$

Express the following in terms of polar coordinates:

1. -3
2. $1 - i$