Lecture 35

Instructor's Comments: The following is a technical note. In the textbook, they use the quadratic formula without any real justification as to why it makes sense with the rest of complex numbers. Here I justify it over real polynomials (and then later we'll make a note that it holds over complex polynomials)

Instructor's Comments: This is a great spot to catch up if you're behind. I would advise grouping the four 'work on their own' problems together at the end of class and spend the last say 15-20 minutes battling through them. Then get students to ask to take one of them up. If doing this I suggest starting with the last problem then mentions the first two in this lecture. The last problem is very easy and they shouldn't have problems. The first two are challenging and you want them to battle through it a bit more. Whatever you get done in class great. Tell them to do the others for homework and refer them to the online notes if they can't solve them.

Example: For $z \in \mathbb{C}$, solve $z^2 - z + 1 = 0$.

Instructor's Comments: Note to students that z will almost exclusively stand for a complex number in this course.

Solution: Ideally, we'd like to write something like

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}.$$

However there is one big gap. The expression $\sqrt{-3}$ has no meaning. Not to mention, we have not discussed what the solutions are to $\sqrt{-3}$ as a complex number. Are there 2 solutions? One solution? Ten solutions? Zero solutions? This needs to be addressed.

Question: What are the solutions to $z^2 = -r$ for $r \in \mathbb{R}$ with $r > 0$?

Solution: Let $z = x + yi$ with $x, y \in \mathbb{R}$. Then

$$-r = z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$$

Therefore, $2xy = 0$ and $x^2 - y^2 = -r$. Thus, either $x = 0$ or $y = 0$. If $y = 0$, then $x^2 = -r$, a contradiction since $x^2 \geq 0$. Hence, $x = 0$ and $-y^2 = -r$ or $y = \pm\sqrt{r}$. Therefore, $z = \pm\sqrt{r}i$. ■

Note: Therefore, we have just validated the use of $\sqrt{-r} = \pm\sqrt{r}i$. The quadratic formula still works for real polynomials (and later we will see it still works for complex polynomials).

Proposition: If the discriminant Δ of a real polynomial $az^2 + bz + c$ is negative, zeroes of the equation are given by

$$z = \frac{-b \pm \sqrt{-\Delta}i}{2a}$$

Solution: We complete the square as usual:

$$\begin{aligned}
a\left(z^2 + \frac{b}{a}z\right) + c &= 0 \\
a\left(z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c &= 0 \\
a\left(z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2\right) - \left(\frac{b^2}{4a}\right) + c &= 0 \\
a\left(z + \frac{b}{2a}\right)^2 &= \left(\frac{b^2}{4a}\right) - c
\end{aligned}$$

Clearing denominators and moving the outer factor in the square term gives

$$(2az + b)^2 = b^2 - 4ac = \Delta$$

Now, since $\Delta < 0$, we can apply the previous result to get that the solutions to this equation are given by

$$\begin{aligned}
2az + b &= \pm\sqrt{-\Delta}i \\
2az &= -b \pm \sqrt{-\Delta}i \\
z &= \frac{-b \pm \sqrt{-\Delta}i}{2a}
\end{aligned}$$

as stated in the claim. ■

The above essentially states that the quadratic formula works as usual if we believe a common convention that $\sqrt{-\Delta}$ is equal to $\sqrt{\Delta}i$ (We shouldn't be writing square roots of negative numbers however!)

Definition: The complex conjugate of a complex number $z = x + yi$ is $\overline{z} := x - yi$.

Instructor's Comments: This is the 13 minute mark

Handout or Document Camera or Class Exercise

Instructor's Comments: Give the next two exercises simultaneously for students to battle through

Solve $z^2 = i\bar{z}$ for $z \in \mathbb{C}$

Solution: Let $z = x + yi$ where $x, y \in \mathbb{R}$. Then

$$\begin{aligned} (x+yi)^2 &= i(x-yi) \\ x^2 - y^2 + 2xyi &= y + xi \\ x^2 - y^2 = y \qquad &\text{and} \qquad 2xy = x \end{aligned}$$

The latter implies that $2xy - x = 0$ and hence $x(2y-1) = 0$. Therefore, either $x = 0$ or $y = \frac{1}{2}$. Substituting into the first equation above gives

$$\begin{aligned} &x = 0 \implies -y^2 = y \implies y^2 + y = 0 \implies y = 0 \text{ or } -1 \\ &y = \tfrac{1}{2} \implies x^2 - (\tfrac{1}{2})^2 = \tfrac{1}{2} \implies x^2 = \tfrac{3}{4} \implies x = \pm\tfrac{\sqrt{3}}{2} \end{aligned}$$

Hence, $z \in \{0, -i, \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{-\sqrt{3}}{2} + \frac{1}{2}i\}$. $\blacksquare$

Handout or Document Camera or Class Exercise

Find a real solution to

$$6z^3 + (1 + 3\sqrt{2}i)z^2 - (11 - 2\sqrt{2}i)z - 6 = 0$$

Solution: Take $z = r \in \mathbb{R}$. Then, if this r is a solution, it must satisfy

$$6r^3 + (1 + 3\sqrt{2}i)r^2 - (11 - 2\sqrt{2}i)r - 6 = 0$$

Expanding and collecting terms gives

$$(6r^3 + r^2 - 11r - 6) + (3\sqrt{2}r^2 + 2\sqrt{2}r)i = 0$$

Therefore, $3\sqrt{2}r^2 + 2\sqrt{2}r = 0$. Factoring gives $\sqrt{2}r(3r+2) = 0$ and thus, $r = 0$ or $r = \frac{-2}{3}$. Since the real part above must also be zero, we see that the r must satisfy

$$6r^3 + r^2 - 11r - 6 = 0$$

Note that $r = 0$ is not a solution to this and that $r = \frac{-2}{3}$ is a solution since

$$6(\tfrac{-2}{3})^3 + (\tfrac{-2}{3})^2 - 11 \cdot \tfrac{-2}{3} - 6 = 6 \cdot \tfrac{-8}{27} + \tfrac{4}{9} + \tfrac{22}{3} - 6 = 0$$

Thus, $r = \frac{-2}{3}$ is the lone solution. ■

Instructor's Comments: This is the 35 minute mark

Proposition: (Properties of Conjugates (PCJ)) Let $z, w \in \mathbb{C}$. Then

(i) $\overline{z+w} = \overline{z} + \overline{w}$

(ii) $\overline{zw} = \overline{z} \cdot \overline{w}$

(iii) $\overline{\overline{z}} = z$

(iv) $z + \overline{z} = 2\Re(z)$

(v) $z - \overline{z} = 2i\Im(z)$.

Instructor's Comments: For your sanity's sake, you should only do a few of these, say 2 and 3.

Solution: Let $z = x + yi$ and $w = u + vi$. Then

(i)

$$\begin{aligned}\overline{z+w} &= \overline{x + yi + u + vi} \\ &= \overline{(x+u) + (y+v)i} \\ &= (x+u) - (y+v)i \\ &= x - yi + u - vi \\ &= \overline{z} + \overline{w}\end{aligned}$$

(ii)

$$\begin{aligned}\overline{zw} &= \overline{(x+yi)(u+vi)} \\ &= \overline{(xu - yv) + (xv + uy)i} \\ &= (xu - yv) - (xv + uy)i \\ &= (x - yi)(u - vi) \\ &= \overline{z}\overline{w}\end{aligned}$$

(iii) $\overline{\overline{z}} = \overline{\overline{x + yi}} = \overline{x - yi} = x + yi = z$

(iv) $z + \overline{z} = x + yi + x - yi = 2x = 2\Re(z)$

(v) $z - \overline{z} = x + yi - (x - yi) = 2yi = 2i\Im(z)$

Instructor's Comments: This is the 40 minute mark.

Handout or Document Camera or Class Exercise

Prove the following for $z \in \mathbb{C}$

(i) $z \in \mathbb{R}$ if and only if $z = \bar{z}$.

(ii) z is purely imaginary if and only if $z = -\bar{z}$.

Instructor's Comments: Note that 0 is both real and purely imaginary.

Solution:

(i) ($\Rightarrow$) Let $z = x + 0i \in \mathbb{R}$. Then $\overline{z} = x - 0i = x = z$.

($\Leftarrow$) Let $z = x + yi$ for $x, y \in \mathbb{R}$. Assume that $z = \overline{z}$. Then,

$$\begin{aligned} z &= \overline{z} \\ x + yi &= x - yi \\ y &= -y \\ 2y &= 0 \\ y &= 0 \end{aligned}$$

Therefore, $z = x + 0i \in \mathbb{R}$. ■

(ii)

$$\begin{aligned} z \text{ is purely imaginary } &\Leftrightarrow iz \in \mathbb{R} \\ &\Leftrightarrow iz = \overline{iz} && \text{By the above} \\ &\Leftrightarrow iz = -i\overline{z} && \text{By PCJ} \\ &\Leftrightarrow z = -\overline{z} \end{aligned}$$

completing the proof. ■

Instructor's Comments: This is the 50 minute mark.