Complex Numbers

Current view $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

These sets can be thought of as helping us to solve polynomial equations.
However $x^2+1=0$ has no solution in <u>any</u> of these sets.

Def'n: A complex number (in standard form) is an expression of the form $x+yi$ where $x, y \in \mathbb{R}$ and i is the imaginary unit.
Denote the set of complex numbers by

$$\mathbb{C} := \{x+yi : x, y \in \mathbb{R}\}.$$

Ex: $1+2i$, $3i$, $\sqrt{13}+\pi i$, $2+0i$

Note: $\mathbb{R} \subseteq \mathbb{C}$.

If $z = x+yi$ then $x = Re(z) = \Re(z)$ real part.
$y = Im(z) = \Im(z)$ imaginary part.

Two complex numbers $z = x + yi$ and $w = u + vi$ are equal iff

$$x = u \quad \& \quad y = v.$$

A complex number z is

- purely real if $\operatorname{Im}(z) = 0$ ie $z = x$
- Purely imaginary if $\operatorname{Re}(z) = 0$ ie $z = yi$.

We turn $\mathbb{C}$ into a ring by defining $+, -, \cdot$ b

$$(x+yi) \pm (u+vi) = (x \pm u) + (y \pm v)i$$

$$(x+yi)(u+vi) = (xu - vy) + (xv + uy)i$$

By this def'n:

$$i^2 = i \cdot i = (0+i)(0+i) = -1 + 0i = -1$$

<u>So</u> i is a solution to $x^2 + 1 = 0$.

With this, multiplication can be remembered b

$$(x+yi)(u+vi) = xu + xvi + uyi + vyi^2$$

$$= xu - vy + (xv + uy)i$$

Ex: $(1+2i)+(3+4i) = 4+6i$

$(1+2i)-(3+4i) = -2-2i$

$(1+2i)(3+4i) = 3-8+(4+6)i$

$= -5+10i$

Commutative

Rings (hence $\mathbb{C}$) have the following properties

1. Associativity (Let $v, w, z \in \mathbb{C}$ & $z = x+yi$)

$(v+w)+z = v+(w+z)$

& $(vw)z = v(wz)$

2. Commutativity

$w+v = v+w$ & $wv = vw$

3. Identities

$z+0 = z$ & $z \cdot 1 = z$ where

$0 = 0+0i$ & $1 = 1+0i$

4. Additive inverses

$z+(-z) = 0$ where $-z = -x-yi$

5. Distributive Property

$$z(w+v) = zw + zv$$

We turn $\mathbb{C}$ into a field by defining the inverse operation for nonzero complex numbers

$$(x+yi)^{-1} := \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$$

Note: If $z \in \mathbb{C}$ & $z \neq 0$ then

$$z \cdot z^{-1} = 1 \quad \text{(Exercise)}$$

For complex numbers u, v, w, z with v, z nonzero, the above is consistent with the usual fraction rules:

$$\frac{u}{v} + \frac{w}{z} = \frac{uz+vw}{vz} \quad \& \quad \frac{u}{v} \cdot \frac{w}{z} = \frac{uw}{vz}$$

For $k \in \mathbb{N}$, $z \in \mathbb{C}$ define $(z \neq 0)$

$$z^0 = 1 \quad z^1 = z \quad \& \quad z^{k+1} = z \cdot z^k$$

Define $z^{-k} := (z^{-1})^k$

Usual exponent rules hold ie

$$(z^m)^n = z^{mn} \quad \& \quad z^m \cdot z^n = z^{m+n}$$

(for $m, n \in \mathbb{Z}$).

Ex: Write $\frac{1+2i}{3-4i}$ in standard form

Sol'n:

$$\begin{aligned}
\frac{1+2i}{3-4i} &= (1+2i)(3-4i)^{-1} \\
&= (1+2i)\left(\frac{3}{3^2+4^2} - \frac{(-4)}{3^2+4^2}i\right) \\
&= (1+2i)\left(\frac{3}{25} + \frac{4}{25}i\right) \\
&= \frac{3}{25} - \frac{8}{25} + \left(\frac{4}{25} + \frac{6}{25}\right)i \\
&= \frac{-5}{25} + \frac{10}{25}i \\
&= \frac{-1}{5} + \frac{2}{5}i
\end{aligned}$$

Express the following in standard form:

1. $\frac{(1-2i)-(3+4i)}{5-6i} = S$

2. $i^{2015} = T$

1. $S = ((1-2i)-(3+4i))(5-6i)^{-1}$

$= (-2-6i)\left(\frac{5}{5^2+6^2} - \frac{(-6)i}{5^2+6^2}\right)$

$= (-2-6i)\left(\frac{5}{61} + \frac{6}{61}i\right)$

$= \left(\frac{-10}{61} + \frac{36}{61}\right) + \left(\frac{-12}{61} - \frac{30}{61}\right)i$

$= \frac{26}{61} - \frac{42}{61}i$

2. $T = i^{2015}$

$i^2 = -1$
$i^4 = 1$

$= (i^4)^{503} \cdot i^3$

$= 1^{503} \cdot i^2 \cdot i$

$= -i = 0 - i$