Lecture 2

Claim: If n is a positive integer, then $n^2 + 1$ is not a perfect square.

Proof: Let n be a positive integer. Then $n^2 < n^2 + 1 < n^2 + 2n + 1 = (n+1)^2$. Since there are no integer squares between n^2 and $(n+1)^2$, we are done. ■

Question: What if we change $n^2 + 1$ to $n^2 + 13$?

Note: When demonstrating this statement, we would need a proof. When showing the statement is false, we need a counterexample.

Solution: This is false. Consider what happens when $n = 6$. Then $n^2 + 13 = 6^2 + 13 = 49 = (7)^2$.

Question: What if we change $n^2 + 1$ to $1141n^2 + 1$?

Solution: This is true for all $n < 10^{24}$. Despite being true for a large number of values, this does not constitute a proof. It turns out in this case this is also false. Consider $n = 30693385322765657197397208$. You can check this in Sage/Python that this does indeed give a counter example (that is, $1141n^2 + 1$ is a perfect square). Interested readers should check out Pell's Equations.

Instructor's Comments: This is the 12 minute mark

Definition: A *statement* is a sentence that is either true or false.

Definition: A *proposition* is a claim that requires a proof.

Definition: A *theorem* is a strong proposition.

Definition: A *lemma* is a weak proposition.

Definition: A *corollary* follows immediately from a proposition.

Definition: An *axiom* is a given truth.

Example: Axiom: The square of a real number is nonnegative.

Example: Axiom: The sum of two even numbers is even. (You could prove this however if you wanted)

Note: In general, axioms are statements that a fellow typical math 135 student should know before entering this class.

Instructor's Comments: This is the 20 minute mark

Example: Show that for $\theta \in \mathbb{R}$, $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$.

Note: $\in$ means 'in; or 'belongs to' and $\mathbb{R}$ is the set of real numbers.

Proof: Recall these three axioms hold for all $x, y \in \mathbb{R}$:

1) $\sin^2(x) + \cos^2(x) = 1$

2) $\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$

3) $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

To prove equalities, we do left hand side to right hand side proofs (or vice versa). We can also meet in the middle and do half starting with the left hand side and half starting with the right hand side.

$$\begin{aligned}
\text{LHS} &= \sin(3\theta) \\
&= \sin(2\theta + \theta) \\
&= \sin(2\theta)\cos(\theta) + \sin(\theta)\cos(2\theta) && \text{Use identity 2) with } x = 2\theta \text{ and } y = \theta \\
&= (2\sin(\theta)\cos(\theta))\cos(\theta) + \sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) && \text{Use identity 2) and 3) with } x = y = \theta \\
&= 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)) \\
&= 3\sin(\theta)(1 - \sin^2(\theta)) - \sin^3(\theta)) && \text{Use identity 1) with } x = \theta \\
&= 3\sin(\theta) - 4\sin^3(\theta) \\
&= \text{RHS}
\end{aligned}$$

Note: Make sure to identify the uses of trigonometric identities above. Be explicit.

Instructor's Comments: This is the 30-33 minute mark

In what follows, we will discuss good and bad proofs of Stewart's Theorem. Try to prove the theorem yourself.

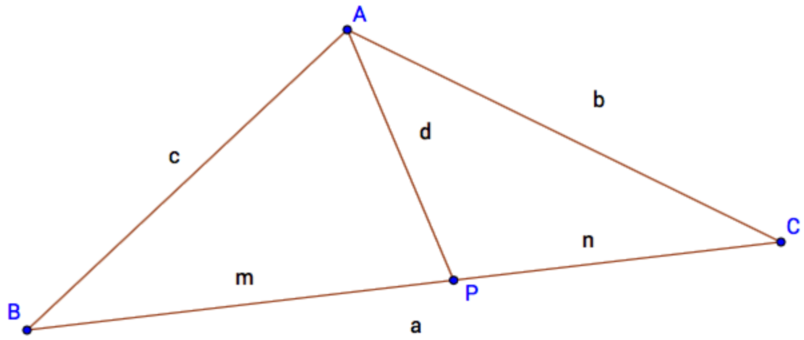
Instructor's Comments: This is the 38 minute mark

Then analyze the proofs for improvement.

Instructor's Comments: This will take you to the 46 minute mark

Handout or Document Camera or Class Exercise

Stewart's Theorem Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$. If P is a point on BC with $BP = m$, $PC = n$ and $AP = d$, then $dad + man = bmb + cnc$.



Proof. **Proof A**

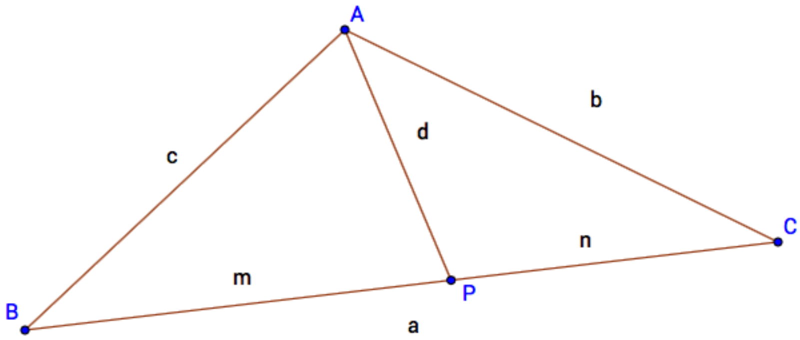
$$c^2 = m^2 + d^2 - 2md\cos\theta$$
$$b^2 = n^2 + d^2 - 2nd\cos\theta'$$
$$b^2 = n^2 + d^2 + 2nd\cos\theta$$
$$\frac{m^2 - c^2 + d^2}{-2md} = \frac{b^2 - n^2 - d^2}{2nd}$$
$$nc^2 - nm^2 - nd^2 = -mb^2 + mn^2 + md^2$$
$$nc^2 - mb^2 = mn^2 + md^2 + nm^2 + nd^2$$
$$cnc + bmb = nm(n + m) + d^2(m + n)$$
$$cnc + bmb = man + dad$$

∎

Note: Unclear what θ and θ' are. No explanation. Division by variables should be careful about 0.

Handout or Document Camera or Class Exercise

Stewart's Theorem Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$. If P is a point on BC with $BP = m$, $PC = n$ and $AP = d$, then $dad + man = bmb + cnc$.



Proof. **Proof B**

The Cosine Law on $\triangle APB$ tells us that

$$c^2 = m^2 + d^2 - 2md\cos(\angle APB).$$

Subtracting c^2 from both sides gives

$$0 = -c^2 + m^2 + d^2 - 2md\cos(\angle APB).$$

Adding $2md\cos\angle APB$ to both sides gives

$$2md\cos(\angle APB) = -c^2 + m^2 + d^2.$$

Dividing both sides by $2md$ gives

$$\cos(\angle APB) = \frac{-c^2 + m^2 + d^2}{2md}.$$

Now, the Cosine Law on $\triangle APC$ tells us that

$$b^2 = n^2 + d^2 - 2nd\cos\angle APC.$$

Since $\angle APC$ and $\angle APB$ are supplementary angles, then

$$\cos\angle APC = \cos(\pi - \angle APB) = -\cos(\angle APB).$$

Substituting into our previous equation, we see that

$$b^2 = n^2 + d^2 + 2nd\cos\angle APB.$$

Subtracting n^2 from both sides gives

$$b^2 - n^2 = d^2 + 2nd\cos(\angle APB).$$

Then subtracting d^2 from both sides gives

$$b^2 - n^2 - d^2 = 2nd\cos(\angle APB).$$

Dividing both sides by $2nd$ gives

$$\frac{b^2 - n^2 - d^2}{2nd} = \cos(\angle APB).$$

Now we have two expressions for $\cos(\angle APB)$ and equate them to yield

$$\frac{-c^2 + m^2 + d^2}{2md} = \frac{b^2 - n^2 - d^2}{2nd}.$$

Multiplying both sides by $2mnd$ shows us that

$$n(-c^2 + m^2 + d^2) = m(b^2 - n^2 - d^2).$$

Next we distribute to get

$$-nc^2 + nm^2 + nd^2 = mb^2 - mn^2 - md^2.$$

Adding $nc^2 + mn^2 + md^2$ to both sides gives

$$nm^2 + mn^2 + nd^2 + md^2 = mb^2 + nc^2.$$

Factoring twice gives:

$$nm(m + n) + d^2(m + n) = mb^2 + nc^2.$$

Since P lies on BC, then $a = m + n$ so we substitute to yield

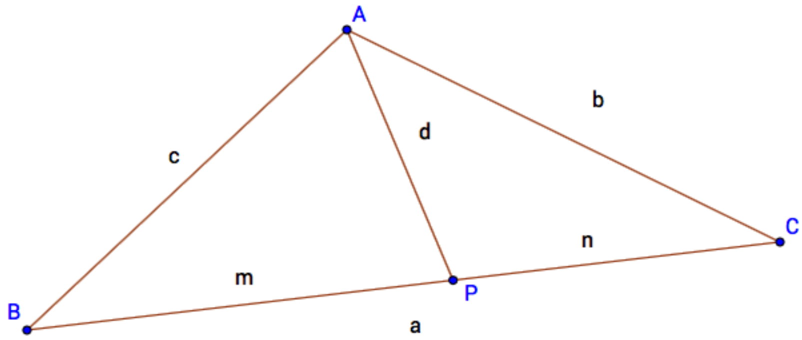
$$nma + d^2a = mb^2 + nc^2.$$

Finally, we can rewrite this as $bmb + cnc = dad + man$.. ■

Note: Too verbose. Can shorten the explanation by not writing out every algebraic manipulation.

Handout or Document Camera or Class Exercise

Stewart's Theorem Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$. If P is a point on BC with $BP = m$, $PC = n$ and $AP = d$, then $dad + man = bmb + cnc$.



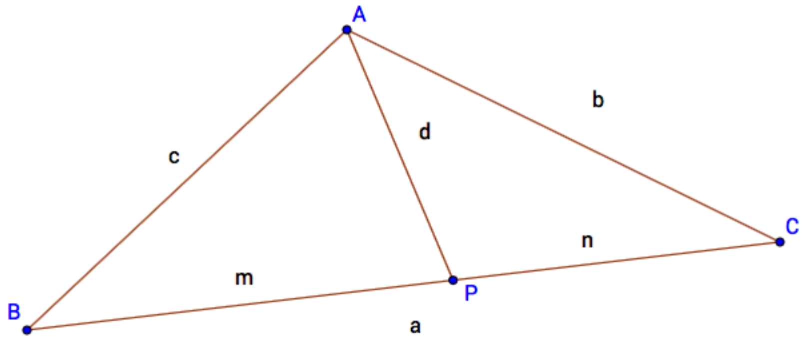
Proof. **Proof C**

Using the Cosine Law for supplementary angles $\angle APB$ and $\angle APC$, and then clearing denominators and simplifying gives $dad + man = bmb + cnc$ as required. ■

Note: No details given. Need to provide some evidence of algebraic manipulation.

Handout or Document Camera or Class Exercise

Stewart's Theorem Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$. If P is a point on BC with $BP = m$, $PC = n$ and $AP = d$, then $dad + man = bmb + cnc$.



Proof. **Proof D**

The Cosine Law on $\triangle APB$ tells us that

$$c^2 = m^2 + d^2 - 2md\cos\angle APB.$$

Similarly, the Cosine Law on $\triangle APC$ tells us that

$$b^2 = n^2 + d^2 - 2nd\cos\angle APC.$$

Since $\angle APC$ and $\angle APB$ are supplementary angles, we have

$$b^2 = n^2 + d^2 + 2nd\cos\angle APB.$$

Equating expressions for $\cos\angle APB$ yields

$$\frac{-c^2 + m^2 + d^2}{2md} = \frac{b^2 - n^2 - d^2}{2nd}.$$

Clearing the denominator and rearranging gives

$$nm^2 + mn^2 + nd^2 + md^2 = mb^2 + nc^2.$$

Factoring yields

$$mn(m+n) + d^2(m+n) = mb^2 + nc^2.$$

Substituting $a = (m+n)$ gives $dad + man = bmb + cnc$ as required. ■

Note: Overall a good proof. Perhaps some more information on why the supplementary angle step holds would be good. Justifying why division by a variable is allowed (that is, nonzero variables) would be a plus and perhaps labeling previous equations to reference in the future would help this proof slightly. This would be an acceptable answer regardless of these minor quibbles.

Instructor's Comments: This concludes up to the 46-48 minute mark

Handout or Document Camera or Class Exercise

Find the flaw in the following arguments:

(i) For $a, b \in \mathbb{R}$,

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= b(a-b) \\ a + b &= b && \text{ERROR: division by 0 since } a = b \\ b + b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

Instructor's Comments: This is the end of lecture 2. Begin Lecture 3 with the next two examples.

(ii)

$$\begin{aligned} x &= \frac{\pi + 3}{2} \\ 2x &= \pi + 3 \\ 2x(\pi - 3) &= (\pi + 3)(\pi - 3) \\ 2\pi x - 6x &= \pi^2 - 9 \\ 9 - 6x &= \pi^2 - 2\pi x \\ 9 - 6x + x^2 &= \pi^2 - 2\pi x + x^2 \\ (3 - x)^2 &= (\pi - x)^2 \\ 3 - x &= \pi - x \\ 3 &= \pi \end{aligned}$$

(iii) For $x \in \mathbb{R}$,

$$\begin{aligned} (x-1)^2 &\geq 0 \\ x^2 - 2x + 1 &\geq 0 \\ x^2 + 1 &\geq 2x \\ x + \tfrac{1}{x} &\geq 2 \end{aligned}$$