Divisibility Rules

L2GP1

A positive integer n is divisible by

(a) 2^k iff the last k digits are divisible by 2^k.

(b) 3 (or 9) iff the sum of the digits is divisible by 3 (or 9)

(c) 5^k iff the last k digits are divisible by 5^k.

(d) 7 (or 11 or 13) iff the alternating sum of triples of digits is divisible by 7 (or 11 or 13)

Ex: $n = 123\,456\,333$

$$333 - 456 + 123 = 0$$

$\therefore$ $7, 11, 13 \mid 0 \overset{(d)}{\Rightarrow} 7, 11, 13 \mid n$.

Proof of (b)

Let $n \in \mathbb{N}$. Write

$$n = d_0 + 10d_1 + 10^2 d_2 + \ldots + 10^k d_k$$

where $d_i \in \{0, 1, \ldots 9\}$

(Ex: $213 = 3 + 10(1) + 100(2)$)

<u>So</u>: $9 \mid n \iff n \equiv 0 \mod 9$

$$\iff 0 \equiv d_0 + 10d_1 + 10^2 d_2 + \ldots + 10^k d_k \mod 9$$

By PC $\iff 0 \equiv d_0 + d_1 + d_2 + \cdots + d_k \mod 9$

Thus, $9 \mid n \iff$ 9 divides the sum of the digits of n. $\blacksquare$

"Random" Examples

$3 \equiv 24 \mod 7$ and $1 \equiv 8 \mod 7$
$3 \equiv 27 \mod 6$ and $1 \not\equiv 9 \mod 6$

Proposition (Congruences & Division - CD)
Let $a, b, c \in \mathbb{Z}$ & $n \in \mathbb{N}$
If $ac \equiv bc \mod n$ and $\gcd(c,n)=1$ then $a \equiv b \mod n$.

Pf: By assumption, $n \mid ac-bc$ so $n \mid c(a-b)$. Since $\gcd(c,n)=1$, by CAD, $n \mid a-b$. Hence
$a \equiv b \mod n$. ⊠

What is the remainder when $77^{100}(999) - 6^{83}$ is divided by 4?

$77 = 19(4) + 1$

$999 = 249(4) + 3$

By CISR, $77 \equiv 1 \mod 4$

$999 \equiv 3 \mod 4$

Thus, by PC

$$77^{100}(999) - 6^{83}$$
$$\equiv (1)^{100}(3) - 2^{83} \mod 4$$
$$\equiv 3 - 2^2 \cdot 2^{81} \mod 4$$
$$\equiv 3 - 4 \cdot 2^{81} \mod 4$$
$$\equiv 3 - 0 \cdot 2^{81} \mod 4$$
$$\equiv 3 \mod 4.$$

By CISR, 3 is the remainder when $77^{100}(999) - 6^{83}$ is divided by 4.

Proposition (Congruent iff Same Remainder CISR). Let $a, b \in \mathbb{Z}$. Then $a \equiv b \bmod n \iff a$ & b have the same remainder after division by n.

Pf: By the Division Algorithm, write

$$a = nq_a + r_a$$
$$b = nq_b + r_b$$

where $0 \leq r_a, r_b < n$. Subtracting

$$a - b = n(q_a - q_b) + r_a - r_b \quad (1)$$

$\Rightarrow$ Assume $a \equiv b \bmod n$ ie. $n \mid a-b$. Since $n \mid n(q_a - q_b)$ by DIC,

$$n \mid r_a - r_b.$$

By our restriction

$$-n+1 \leq r_a - r_b \leq n-1$$

BUT only 0 is divisible by n in this range! Since $n \mid r_a - r_b$, we must have that $r_a - r_b = 0$. Hence $r_a = r_b$.

$\Leftarrow$ Assume $r_a = r_b$. By (1),

$$a - b = n(q_a - q_b)$$

$$\Rightarrow n \mid a - b \Rightarrow a \equiv b \mod n. \ \blacksquare$$

What is the last digit of $5^{32}3^{10} + 9^{22}$?