LDET2

Let $d = \gcd(a,b)$ where $a \neq 0$ and $b \neq 0$. If $(x,y) = (x_0, y_0)$ is a solution to the LDE

$$ax + by = c$$

Then all solutions are given by

$$x = x_0 + \frac{b}{d}n \qquad y = y_0 - \frac{a}{d}n$$

for all $n \in \mathbb{Z}$. (Alternatively:

$$\{(x_0 + \tfrac{b}{d}n,\ y_0 - \tfrac{a}{d}n) : n \in \mathbb{Z}\}.$$

Pf: Note the above are actually solutions to the LDE.

Now, let (x,y) be another solution to the LDE. Thus

$$ax + by = c$$

$$\underline{ax_0 + by_0 = c}$$

Subtract: $a(x - x_0) + b(y - y_0) = 0$

$$a(x-x_0) = -b(y-y_0)$$

$$\frac{a}{d}(x-x_0) = -\frac{b}{d}(y-y_0) \quad (1)$$

Now, since $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$ (by DBGCD) ~~we have that~~ and since

$$\frac{b}{d} \mid -\frac{b}{d}(y-y_0) = \frac{a}{d}(x-x_0)$$

we use CAD to see that $\frac{b}{d} \mid x-x_0$.
Thus, $\exists n \in \mathbb{Z}$ s.t. $x-x_0 = \frac{b}{d}n$ and thus $x = x_0 + \frac{b}{d}n$. Plug into (1):

$$\frac{a}{d}\left(\frac{b}{d}n\right) = -\frac{b}{d}(y-y_0)$$

$$-\frac{a}{d}n = y-y_0$$

$$\Rightarrow y = y_0 - \frac{a}{d}n.$$

Q: Alice has a lot of mail to send. She wishes to spend exactly 100 dollars buying 49-cent & 53-cent stamps. In how many ways can she do this?

Sol'n: Let x be the number of 49 cent stamps. Let y be the number of 53 cent stamps. Note $x, y \in \mathbb{Z}$ and $x, y \geq 0$.

WANT to solve

$$0.49x + 0.53y = 100$$
$$49x + 53y = 10000$$

x	y	r	q
0	1	53	0
1	0	49	0
-1	1	4	1
13	-12	1	12
		0	4.

$\therefore$ $49(13) + 53(-12) = 1$

$$49(130000) + 53(-120000) = 10000$$

Thus, by LDET2

$$x = 130000 - 53n \qquad \forall n \in \mathbb{Z}$$
$$y = -120000 + 49n$$

Since $x \geq 0$, we have

$$130000 - 53n \geq 0$$
$$2452 + \frac{44}{53} = \frac{130000}{53} \geq n$$

Since $y \geq 0$, we have

$$-120000 + 49n \geq 0$$
$$\Rightarrow \quad n \geq \frac{120000}{49} = 2448 + \frac{48}{49}$$

Since $n \in \mathbb{Z}$,

$$2449 \leq n \leq 2452$$

Thus, there are 4 possible solutions. $\blacksquare$

Find all non-negative integer solutions to $15x - 24y = 9$ where $x \leq 20$ and $y \leq 20$.

$\div 3 \qquad 5x - 8y = 3$

Note $x_0 = -1$ & $y_0 = -1$ is a solution.

Since $\gcd(5, -8) = 1$, by LDET2.

$$x = -1 - (-8)n = -1 + 8n$$
$$y = -1 + 5n \quad = -1 + 5n \qquad \forall n \in \mathbb{Z}$$

is the solution set. By the statement

$0 \leq x \leq 20$ & $0 \leq y \leq 20$

$0 \leq -1 + 8n \leq 20$ & $0 \leq -1 + 5n \leq 20$

$1 \leq 8n \leq 21$ & $1 \leq 5n \leq 21$

$\Rightarrow n = 1, 2$ $\qquad$ $\Rightarrow n = 1, 2, 3, 4$

Thus, $n = 1, 2$ gives the only solutions of

$(7, 4)$ & $(15, 9)$ $\blacksquare$.

Congruences.

Idea: Simplify problems in Divisibility.

Q: Is 156723 divisible by 11?

What angle do you get after a 1240° rotation?

What time is it 40 hours from now

Idea: We only care about the above answers up to multiples of 11, 360, and 24.

Def'n: Let $m \in \mathbb{N}$. We say that two integers a, b are congruent modulo m iff $m \mid (a-b)$ and we write

$$a \equiv b \bmod m$$

<u>or</u>

$$a \equiv b \pmod{m}$$

If $m \nmid a-b$ we write $a \not\equiv b \bmod m$.

Ex: $7 \equiv 4 \mod 3$ $\quad 7 \not\equiv 4 \mod 4$

$4 \equiv 7 \mod 3$ $\quad 2 \equiv 2 \mod 3$

$10 \equiv 15 \mod 5$ & $15 \equiv 30 \mod 5$

& $10 \equiv 30 \mod 5$.