L16P01

CODE

BC

Q1. I enjoy trying to discover and write MATH 135 proofs.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q2. When I have difficulties with MATH 135 proofs, I know I can handle them.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q3. A statement $P(n)$ is proved true for all $n \in \mathbb{N}$ by induction.

In this proof, for some natural number k, we might:

A) Prove $P(1)$. Prove $P(k)$. Prove $P(k+1)$.

B) Assume $P(1)$. Prove $P(k)$. Prove $P(k+1)$.

C) Prove $P(1)$. Assume $P(k)$. Prove $P(k+1)$.

D) Prove $P(1)$. Assume $P(k)$. Assume $P(k+1)$.

E) Assume $P(1)$. Prove $P(k)$. Assume $P(k+1)$.

Find a closed form expression for $\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)$.

Solution: Last class, we hypothesized that the product above is equal to $\frac{n+1}{2n}$. Let $P(n)$ be the statement that

$$\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)=\frac{n+1}{2n}.$$

We prove $P(n)$ is true for all values of $n \geq 2$, by induction.

Base Case: $n=2$

$$\prod_{r=2}^{2}\left(1-\frac{1}{r^2}\right) = 1-\frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2(2)} \checkmark$$

IH: $P(k)$ is true for some $k \geq 2$, $k \in \mathbb{N}$.

$$\prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right) = \frac{k+1}{2k}.$$

IStep: WANT $\prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right) = \frac{(k+1)+1}{2(k+1)}$

$$\prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right) = \prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right)$$

$$\overset{IH}{=} \frac{\cancel{k+1}}{2k}\cdot\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}$$

$$= \frac{k^2+2k+\cancel{1}-\cancel{1}}{2k(k+1)}$$

$$= \frac{\cancel{k}(k+2)}{2\cancel{k}(k+1)}$$

$$= \frac{k+2}{2(k+1)} = \text{RHS}.$$

$\therefore P(k+1)$ is true.

$\therefore P(n)$ is true $\forall n \in \mathbb{N}$ & $n \geq 2$ by POMI.

$\square$

L16 P4

Examine the following induction "proofs". Find the mistake

Question: For all $n \in \mathbb{N}$, $n > n + 1$.

Proof: Let $P(n)$ be the statement: $n > n + 1$. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $k > k + 1$ for some integer $k \geq 1$. We must show that $P(k + 1)$ is true, that is, $k + 1 > k + 2$. But this follows immediately by adding one to both sides of $k > k + 1$. Since the result is true for $n = k + 1$, it holds for all n by the Principle of Mathematical Induction. NO BASE CASE.

Question: All horses have the same colour. (Cohen 1961).

Proof:

Base Case: If there is only one horse, there is only one colour.

Inductive hypothesis and step: Assume the induction hypothesis that within any set of n horses for any $n \in \mathbb{N}$, there is only one colour. Now look at any set of $n + 1$ horses. Number them: $1, 2, 3, ..., n, n + 1$. Consider the sets $\{1, 2, 3, ..., n\}$ and $\{2, 3, 4, ..., n + 1\}$. Each is a set of only n horses, therefore by the induction hypothesis, there is only one colour. But the two sets overlap, so there must be only one colour among all $n + 1$ horses. FALSE when $n=1$!

Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be factored "uniquely" as a product of primes up to reordering.

Pf: Existence

Assume towards a contradiction that not every number can be factored into primes. Let n be the smallest such number (Well Ordering Principle). Either n is prime # OR $n = ab$ with $1 < a, b < n$. However, since $a, b < n$, a & b can be written as a product of primes. Thus, $n = ab$ is a product of primes, contradicting the def'n of n.

Uniqueness

Assume towards a contradiction that $\exists\, n>1, n\in\mathbb{N}$ s.t.

$$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_m$$

By def'n $p_1 \mid n = q_1 \cdots q_m$. Thus, $p_1 \mid q_j$ for some $1 \leq j \leq m$. Thus $p_1 = q_j$. WLOG assume $j=1$. So $p_1 = q_1$ (otherwise rearrange).

Then, $p_2 \cdots p_k = q_2 \cdots q_m$. Take n to be minimal (Well ordering Principle)

As $p_2 \cdots p_k < n$ and $q_2 \cdots q_m < n$,

Thus, $k=m$ and $p_i = q_j$ in some order

Thus, $p_2 \cdots p_k = q_2 \cdots q_k$ (upto reordering)

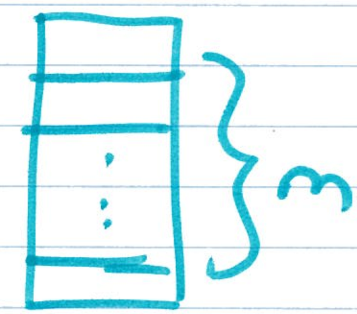
$$p_1 p_2 \cdots p_k = p_1 q_2 \cdots q_k = q_1 q_2 \cdots q_k.$$

This contradicts the existence of n. $\boxed{}$.

Q: Exactly $mn-1$ breaks are always needed to break a $m \times n$ chocolate rectangle into unit squares.

Pf: Fix $m \in \mathbb{N}$. Use induction on n.

Base Case $n=1$



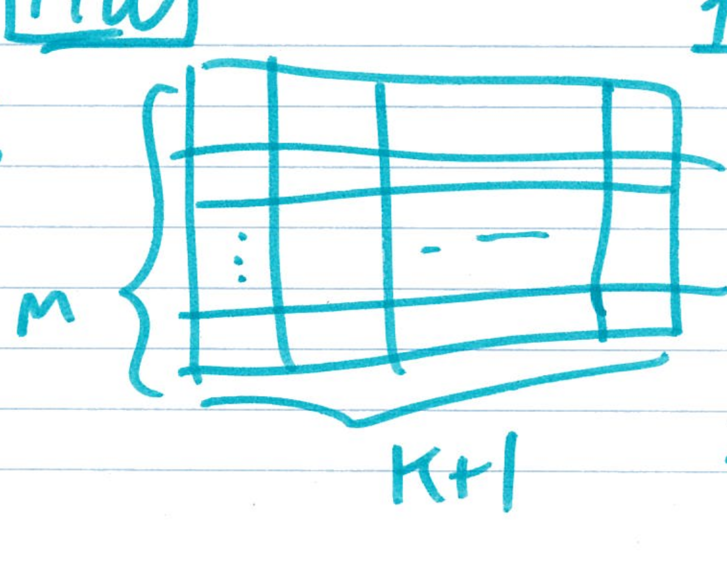
Takes $m-1$ cracks. ✓

$= m(1)-1$

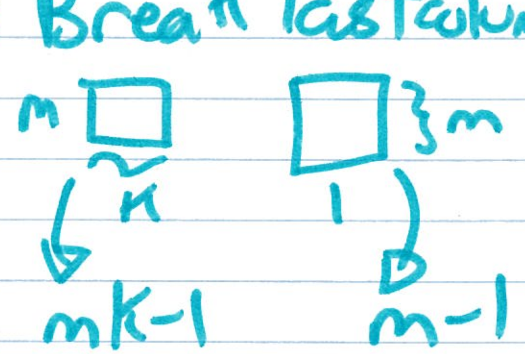
$= mn-1$

IH: $\boxed{\text{HW}}$

I Step:



1: Break last column



$1 + mk - 1 + m - 1$

$= m(k+1) - 1$. ∎