Lecture 14

Handout or Document Camera or Class Exercise

Instructor's Comments: In this lecture, you might want to consider giving a midterm survey of your teaching.

Prove $P(n) : 6 \mid (2n^3 + 3n^2 + n)$ holds $\forall n \in \mathbb{N}$.

Solution:

(i) Base case

$$2n^3 + 3n^2 + n = 2 + 3 + 1 = 6$$

and $6 \mid 6$. Hence $P(1)$ is true.

(ii) Inductive Hypothesis. Assume $P(k)$ is true for some $k \in \mathbb{N}$, that is, $\exists \ell \in \mathbb{Z}$ such that $6\ell = 2k^3 + 3k^2 + k$.

(iii) Inductive Step: Prove that $P(k+1)$ is true.

$$\begin{aligned} 2(k+1)^3 + 3(k+1)^2 + (k+1) &= 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 + k + 1 \\ &= (2k^3 + 3k^2 + k) + 6k^2 + 12k + 6 \\ &= 6\ell + 6(k^2 + 2k + 1) && \text{IH} \\ &= 6(\ell + (k+1)^2) \end{aligned}$$

Hence, $6 \mid 2(k+1)^3 + 3(k+1)^2 + (k+1)$. Thus $P(k+1)$ is true. Hence by the Principle of Mathematical Induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$. ■

Instructor's Comments: This is the 10 minute mark

Handout or Document Camera or Class Exercise

Instructor's Comments: This illustrates the need for something "stronger" than induction.

Let $\{x_n\}$ be a sequence defined by $x_1 = 4$, $x_2 = 68$ and

$$x_m = 2x_{m-1} + 15x_{m-2} \qquad \text{for all } m \geq 3$$

Prove that $x_n = 2(-3)^n + 10 \cdot 5^{n-1}$ for $n \geq 1$.

Solution: We proceed by induction.

Base Case: For $n = 1$, we have

$$x_1 = 4 = 2(-3)^1 + 10 \cdot 5^0 = 2(-3)^n + 10 \cdot 5^{n-1}.$$

Inductive Hypothesis: Assume that

$$x_k = 2(-3)^k + 10 \cdot 5^{k-1}$$

is true for some $k \in \mathbb{N}$.

Inductive Step: Now, for $k + 1$,

$$\begin{aligned} x_{k+1} &= 2x_k + 15x_{k-1} && \text{Only true if } k \geq 2!!! \\ &= 2(2(-3)^k + 10 \cdot 5^{k-1}) + 15x_{k-1} \\ &= 4(-3)^k + 20 \cdot 5^{k-1} + 15x_{k-1} \\ &= ...? \end{aligned}$$

Instructor's Comments: This is the 15 minute mark

Principle of Strong Induction (POSI)

Axiom: If sequence of statements $P(1)$, $P(2)$, ... satisfy

(i) $P(1) \wedge P(2) \wedge ... \wedge P(b)$ are true for some $b \in \mathbb{N}$

(ii) $P(1) \wedge P(2) \wedge ... \wedge P(k)$ are true implies that $P(k+1)$ is true for all $k \in \mathbb{N}$

then $P(n)$ is true for all $n \in \mathbb{N}$.

Note: This is equivalent in strength to the Principle of Mathematical Induction and to the Well Ordering Principle!

Question: Let $\{x_n\}$ be a sequence defined by $x_1 = 4$, $x_2 = 68$ and

$$x_m = 2x_{m-1} + 15x_{m-2} \qquad \text{for all } m \geq 3$$

Prove that $x_n = 2(-3)^n + 10 \cdot 5^{n-1}$ for $n \geq 1$.

Solution: We proceed by strong induction.

Base Case: For $n = 1$, we have

$$x_1 = 4 = 2(-3)^1 + 10 \cdot 5^0 = 2(-3)^n + 10 \cdot 5^{n-1}.$$

For $n = 2$, we have $x_2 = 68$ and

$$2(-3)^2 + 10 \cdot 5^{2-1} = 18 + 50 = 68.$$

Inductive Hypothesis: Assume that

$$x_i = 2(-3)^i + 10 \cdot 5^{i-1}$$

is true for all $i \in \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$ and $k \geq 2$.

Inductive Step: Now, for $k+1$,

$$\begin{aligned}
x_{k+1} &= 2x_k + 15x_{k-1} && \text{Valid since } k \geq 2 \\
&= 2(2(-3)^k + 10 \cdot 5^{k-1}) + 15(2(-3)^{k-1} + 10 \cdot 5^{k-2}) \\
&= 4(-3)^k + 20 \cdot 5^{k-1} + 30(-3)^{k-1} + 150 \cdot 5^{k-2} \\
&= -12(-3)^{k-1} + 100 \cdot 5^{k-2} + 30(-3)^{k-1} + 150 \cdot 5^{k-2} \\
&= 18(-3)^{k-1} + 250 \cdot 5^{k-2} \\
&= 2 \cdot (-3)^2(-3)^{k-1} + 10 \cdot 5^2 \cdot 5^{k-2} \\
&= 2(-3)^{k+1} + 10 \cdot 5^k
\end{aligned}$$

Hence, $x_{k+1} = 2(-3)^{k+1} + 10 \cdot 5^k$. Thus, by the Principle of Strong Induction, we have that $x_n = 2(-3)^n + 10 \cdot 5^{n-1}$ for all $n \geq 1$. ■

Instructor's Comments: This is the 40 minute mark

Handout or Document Camera or Class Exercise

Instructor's Comments: I would make them do half of this example - say the base cases and the inductive hypothesis

Suppose $x_1 = 3$, $x_2 = 5$ and for all $m \geq 3$,

$$x_m = 3x_{m-1} + 2x_{m-2}.$$

Prove that $x_n < 4^n$ for all $n \in \mathbb{N}$.

Proof: Let $P(n)$ be the given statement. We prove $P(n)$ by strong induction.

(i) Base cases: $P(1)$ is true since $x_1 = 3 < 4$ and $P(2)$ is true since $x_2 = 5 < 16 = 4^2$.

(ii) Inductive Hypothesis: Assume that $P(i)$ is true for all $i \in \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$ with $k \geq 2$.

(iii) Inductive Step. For $k \geq 2$, we have

$$\begin{aligned} x_{k+1} &= 3x_k + 2x_{k-1} && \text{Valid since } k+1 \geq 3 \\ &< 3 \cdot 4^k + 2 \cdot 4^{k-1} \\ &< 4^{k-1}(3 \cdot 4 + 2) \\ &= 4^{k-1}(14) \\ &< 4^{k-1}(16) \\ &= 4^{k+1} \end{aligned}$$

Hence $P(k+1)$ is true and thus $P(n)$ is true for all $n \in \mathbb{N}$ by the Principle of Strong Induction.

Fibonacci Sequence Definition: Define a sequence by $f_1 = 1$, $f_2 = 1$ and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{For all } n \geq 3$$

so $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, and so on.

Note: For a cool link between this sequence and music, check out Tool - Lateralus on Youtube!

Instructor's Comments: This is the 50 minute mark