Ex: Prove $P(n): 6 \mid 2n^3+3n^2+n \ \forall n \in \mathbb{N}$.

Base Case: $n=1$

$2n^3+3n^2+n = 2+3+1 = 6$ and $6 \mid 6$ ✓.

Induction Hypothesis (IH):

Assume $P(k)$ is true for some $k \in \mathbb{N}$.

ie. $\exists \ell \in \mathbb{Z}$ s.t. $6\ell = 2k^3+3k^2+k$.

Inductive Step: Prove $P(k+1)$ is true.

$$2(k+1)^3+3(k+1)^2+(k+1)$$
$$= 2k^3+6k^2+6k+2+3k^2+6k+3+k+1$$
$$= \underbrace{2k^3+3k^2+k}+6k^2+12k+6.$$
$$\overset{IH}{=} 6\ell+6k^2+12k+6$$
$$= 6(\underbrace{\ell+k^2+2k+1}_{\in \mathbb{Z}}). \quad \therefore 6 \mid 2(k+1)^3+3(k+1)^2+(k+1)$$

$\therefore P(k+1)$ is true

So, by POMI, $P(n)$ is true $\forall n \in \mathbb{N}$. ∎

L14P2

Let $\{x_n\}$ be a sequence defined by $x_1 = 4$, $x_2 = 68$ and

$$x_m = 2x_{m-1} + 15x_{m-2} \qquad \text{for all } m \geq 3$$

Prove that $x_n = 2(-3)^n + 10 \cdot 5^{n-1}$ for $n \geq 1$.

Solution: We proceed by induction.

Base Case: For $n = 1$, we have

$$x_1 = 4 = 2(-3)^1 + 10 \cdot 5^0 = 2(-3)^n + 10 \cdot 5^{n-1}.$$

Inductive Hypothesis: Assume that

$$x_k = 2(-3)^k + 10 \cdot 5^{k-1}$$

is true for some $k \in \mathbb{N}$.

Inductive Step: Now, for $k + 1$,

$$\begin{aligned} x_{k+1} &= 2x_k + 15x_{k-1} \\ &= 2(2(-3)^k + 10 \cdot 5^{k-1}) + 15x_{k-1} \\ &= 4(-3)^k + 20 \cdot 5^{k-1} + 15x_{k-1} \\ &= \ldots? \end{aligned}$$

Principle of Strong Induction.

Let $P(n)$ be a statement. If

(i) $P(1), P(2), \ldots, P(b)$ are true for some $b \in \mathbb{N}$

(ii) $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ true $\Rightarrow P(k+1)$ is true $\forall k \in \mathbb{N}$

Then $P(n)$ is true $\forall n \in \mathbb{N}$.

Q: Let $\{x_n\}$ be a sequence s.t. $x_1 = 4$, $x_2 = 68$ and

$$x_n = 2x_{n-1} + 15x_{n-2} \quad \forall n \geq 3.$$

Prove $P(n)$: $x_n = 2(-3)^n + 10 \cdot 5^{n-1} \quad \forall n \geq 1$

Pf: Base Cases.

$n=1$ $\quad x_1 = 4 = 2(-3)^1 + 10 \cdot 5^{1-1} = 2(-3)^n + 10 \cdot 5^{n-1}$

$n=2$ $\quad x_2 = 68$ & $2(-3)^2 + 10 \cdot 5^{2-1} = 18 + 50 = 68.$

IH: $P(i)$ is true for all $i \in \{1, 2, \ldots k\}$ for some $k \in \mathbb{N}$ $(k \geq 2)$.

I Step: For $k \in \mathbb{N}$ with $k \geq 2$,

$$x_{k+1} = 2x_k + 15x_{k-1} \quad (\because k+1 \geq 3)$$

$$\overset{IH}{=} 2\left(2(-3)^k + 10\cdot 5^{k-1}\right) + 15\left(2(-3)^{k-1} + 10\cdot 5^{k-2}\right)$$

$$= 4(-3)^k + 20\cdot 5^{k-1} + 30(-3)^{k-1} + 150\cdot 5^{k-2}$$

$$= (-3)^{k-1}(-12+30) + 5^{k-2}(100+150)$$

$$= (-3)^{k-1}(18) + 5^{k-2}(250)$$

$$= (-3)^{k-1}\left(2\cdot(-3)^2\right) + 5^{k-2}(5^2\cdot 10)$$

$$= 2\cdot(-3)^{k+1} + 10\cdot 5^k$$

Thus $P(k+1)$ is true.

Hence $P(n)$ is true $\forall n \in \mathbb{N}$ by POSI. $\blacksquare$

Suppose $x_1 = 3$, $x_2 = 5$ and

$$x_m = 3x_{m-1} + 2x_{m-2} \quad \forall m \geq 3.$$

Prove $x_n < 4^n$ $\forall n \in \mathbb{N}$.

Pf: Let $P(n)$ be the given statement.
We prove $P(n)$ by strong induction.

Base cases:

$n=1$ $\quad x_1 = 3 < 4$ $\qquad n=2$ $\quad x_2 = 5 < 16 = 4^2$

IH: Assume $P(i)$ is true forall
$i \in \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$ $(k \geq 2)$.

I. Step: For $k \geq 2$,

$$\begin{aligned} x_{k+1} &= 3x_k + 2x_{k-1} \\ &\overset{\text{IH}}{<} 3 \cdot 4^k + 2 \cdot 4^{k-1} \\ &= 4^{k-1}(3 \cdot 4 + 2) \end{aligned}$$

*

$$= 4^{k-1}(14)$$
$$< 4^{k-1} \cdot 16$$
$$= 4^{k+1}.$$

$\therefore P(k+1)$ is true. Thus, $P(n)$ is true $\forall n \in \mathbb{N}$. ∎

Fibonacci Sequence

Define a sequence

$f_1 = 1 \quad f_2 = 1$ and

$f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 3.$

So, $f_3 = 2, \; f_4 = 3, \; f_5 = 5, \ldots$

Tool - Lateralus