Lecture 13

Principle of Mathematical Induction (POMI)

Axiom: If sequence of statements $P(1)$, $P(2)$, ... satisfy

(i) $P(1)$ is true

(ii) For any $k \in \mathbb{N}$, if $P(k)$ is true then $P(k+1)$ is true

then $P(n)$ is true for all $n \in \mathbb{N}$.

Instructor's Comments: Here describe the domino analogy. Explain that you're creating a chain of implications $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$, and so on and you want the chain to begin.

In practice, these arguments proceed as follows:

(i) Prove the base case, that is, verify that $P(1)$ is true

(ii) Inductive hypothesis: Let $k \in \mathbb{N}$ be an arbitrary number. Assume that $P(k)$ is true.

(iii) Inductive conclusion. Deduce that $P(k+1)$ is true.

(iv) Then conclude by the Principle of Mathematical Induction (POMI) that $P(n)$ holds

Instructor's Comments: Emphasize the for some part in the IH step. Note also that the induction proof needn't start at 1 (it could start at 0 or -1 etc.)

Example: Prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in \mathbb{N}$.

Proof: Let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

holds. We prove $P(n)$ is true for all natural numbers n by the Principle of Mathematical Induction.

(i) Base case: When $n = 1$, $P(1)$ is the statement that

$$\sum_{i=1}^{1} i^2 = \frac{(1)((1)+1)(2(1)+1)}{6}.$$

This holds since

$$\frac{(1)((1)+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1 = \sum_{i=1}^{1} i^2.$$

(ii) Inductive Hypothesis. Assume that $P(k)$ is true for some $k \in \mathbb{N}$. This means that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

(iii) Inductive Step. We now need to show that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

To do this, we will start with the left hand side, reduce to the assumption made in the inductive hypothesis and then conclude the right hand side.

$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{k+1} i^2 \\
&= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Inductive Hypothesis} \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\left(\frac{2k^2+k}{6} + \frac{6k+6}{6}\right) \\
&= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \text{RHS}
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for all natural numbers n by the Principle of Mathematical Induction. ■

Instructor's Comments: It is important to note where you used the inductive hypothesis!

Note: Now, we can finally solve the Tower of Hanoi example for the 100 level tower:

$$\begin{aligned}
V_{tower} &= \sum_{i=1}^{100} V_i \\
&= \sum_{i=1}^{100} \pi i^2(1) \\
&= \pi \sum_{i=1}^{100} i^2 \\
&= \pi \frac{(100)(101)(2(100)+1)}{6} \\
&= 338350\pi
\end{aligned}$$

Instructor's Comments: This could easily be 25-30 minutes of your lecture. The rest of the time is spent doing examples:

Handout or Document Camera or Class Exercise

Prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

holds for all natural numbers n.

Solution:

(i) Base case:

$$\frac{(1)(1+1)}{2} = 1 = \sum_{i=1}^{n} i.$$

(ii) Inductive Hypothesis. Assume that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

holds for some $k \in \mathbb{N}$

(iii) Inductive step. For $k+1$,

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{Inductive Hypothesis} \\
&= (k+1)(\tfrac{k}{2}+1) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Therefore, the claim holds by the Principle of Mathematical Induction for all $n \in \mathbb{N}$. ■

Instructor's Comments: This is the 40 minute mark

Instructor's Comments: An example where we don't start at 1

Example: Prove that $n! > 2^n$ for all $n \in \mathbb{N}$ with $n \geq 4$.

Proof: We proceed by mathematical induction.

(i) Base case: When $n = 4$, notice that $4! = 24 > 16 = 2^4$ so the inequality holds in this case.

(ii) Inductive Hypothesis: Assume that $k! > 2^k$ for some $k \in \mathbb{N}$ with $k \geq 4$.

(iii) Inductive Step: Notice that

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)2^k && \text{Inductive Hypothesis} \\
&> (1+1)2^k && \text{Since } k \geq 4 > 1 \\
&= 2^{k+1}
\end{aligned}$$

Thus, the conclusion holds for all $k \in \mathbb{N}$ with $k \geq 4$ by the Principle of Mathematical Induction. ■

Handout or Document Camera or Class Exercise

Examine the following induction "proofs". Find the mistake

Question: For all $n \in \mathbb{N}$, $n > n + 1$.

Proof: Let $P(n)$ be the statement: $n > n + 1$. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $k > k + 1$ for some integer $k \geq 1$. We must show that $P(k + 1)$ is true, that is, $k + 1 > k + 2$. But this follows immediately by adding one to both sides of $k > k + 1$. Since the result is true for $n = k + 1$, it holds for all n by the Principle of Mathematical Induction.

Instructor's Comments: No base cases!

Question: All horses have the same colour. (Cohen 1961).

Proof:

Base Case: If there is only one horse, there is only one colour.

Inductive hypothesis and step: Assume the induction hypothesis that within any set of n horses for any $n \in \mathbb{N}$, there is only one colour. Now look at any set of $n+1$ horses. Number them: $1, 2, 3, ..., n, n + 1$. Consider the sets $\{1, 2, 3, ..., n\}$ and $\{2, 3, 4, ..., n + 1\}$. Each is a set of only n horses, therefore by the induction hypothesis, there is only one colour. But the two sets overlap, so there must be only one colour among all $n+1$ horses.

Instructor's Comments: However, the logic of the inductive step is incorrect for n = 1, because the statement that "the two sets overlap" is false (there are only n + 1 = 2 horses prior to either removal, and after removal the sets of one horse each do not overlap. This is the 50 minute mark