principle of mathematical induction (form 1)

Recall sum & product notation from Friday:

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \dots + n^3$$

$$\prod_{j=n}^{2n} \frac{1}{j-1} = \frac{1}{n-1} \cdot \frac{1}{n+1-1} \cdots \frac{1}{2n-1} \text{ .. etc}$$

"Factorial" $0! = 1, n! = n \times (n-1)! = n(n-1)\dots 2(1)$
$\{n \geq 1\}$

A sequence $P(1), P(2)\dots$ are true if

(i) ↳ $P(1)$ is true

(ii) ↳ For any $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

(P_1) is true by (i)

$(P_1) \Rightarrow P(2)$ is true by (ii) w/ $(k=1)$

$\therefore P(2)$ is true

$P(2) \Rightarrow P(3)$ by (ii) w/ $(k=2)$

$\therefore P(3)$ is true

In practice, induction argument proceeds as follows:

1. base case: verify $P(1)$ is true
2. inductive hypothesis: Let $k \in \mathbb{N}$ be ARBitrary, Assume $P(k)$ is true
3. ind conclusion deduce $P(k+1)$ is true.

$\therefore$ by pom 1. $P(n)$ holds $\forall n \in \mathbb{N}$

1. $\forall n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$

2. $\forall n \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1) \cdots (P(n))$$

Base case:

For $n=1$, Have

$$\sum_{j=1}^{1} j^2 = 1^2 = 1 = \frac{1}{6}(1)(1+1)(2(1)+1) \checkmark \therefore P(1) \text{ holds}$$

IH: Let k be arbitrary

Assume $P(k)$ holds. I.E.

$$\sum_{j=1}^{k} j^2 = \frac{1}{6} k(k+1)(2k+1) \cdots (1+1)$$

ind conclusion:

we want to show $P(k+1)$ holds, i.e.

$$\sum_{j=1}^{k+1} j^2 = \frac{1}{6}(k+1)(k+2)(2k+3) \triangleleft$$

Now,

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2$$

|| (IH)

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \quad \text{By inductive hypothesis}$$

$$= (k+1)\left[\frac{1}{6} k(2k+1) + (k+1)\right]$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) \triangleleft$$

principle of mathematical induction

∴ Hence $P(k+1)$. holds

By POMI $P(n)$ holds $\forall n \in \mathbb{N}$

3 $\forall n \in \mathbb{N}$

$$\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n} \qquad (P(n))$$

<u>Base case</u>, For $n = 1$, have

$$\prod_{j=2}^{1} \left(1 - \frac{1}{j^2}\right) = 1 = \frac{1+1}{2 \times 1}$$

so $P(1)$ holds.

<u>IH</u> Let $k \in \mathbb{N}$ be Arbitrary, assume $P(k)$ holds. I.E

$$\prod_{j=2}^{k} \left(1 - \frac{1}{j^2}\right) = \frac{k+1}{2k} \qquad \ldots (IH)$$

Induction conclusion: want to show $P(k+1)$ holds I.E

$$\prod_{j=2}^{k+1} \left(1 - \frac{1}{j^2}\right) = \frac{k+2}{2(k+1)}$$

want to be the same

Now

$$\prod_{j=2}^{k+1} \left(1 - \frac{1}{j^2}\right) = \underbrace{\left[\prod_{j=2}^{k} \left(1 - \frac{1}{j^2}\right)\right]}_{IH} \cdot \left[1 - \frac{1}{(k+1)^2}\right]$$

$$= \left(\frac{k+1}{2k}\right) \times \left[1 - \frac{1}{(k+1)^2}\right]$$

By IH

$$\vdots$$

$$\frac{k+2}{2(k+1)}$$

∴ Hence $P(k+1)$ holds.

4. $\forall n \in \mathbb{N},\ n \geq 4,$

$n! > 2^n \quad \cdots \quad P(n)$

Base case: For $n = 4$,

$4! = 24$

$2^4 = 16$

$24 > 16 = 2^4$ ✓

$\therefore$ $P(4)$ holds

IH Let $k \in \mathbb{N}$, $k \geq 4$. Assume $P(k)$ holds.

I.E. $k! > 2^k \quad \cdots \quad (\text{IH})$

Ind Conc We want to show $P(k+1)$ holds, I.E.

$(k+1)! > 2^{k+1}$.

Now,

$$(k+1)! = (k+1)\,\underset{\substack{\vee \\ 2^k}}{k!}$$

$\left[\begin{array}{l} a, b, c \geq 0 \\ b \geq c \\ ab \geq ac \end{array}\right]$

$> (k+1)\, 2^k$ ~~By (IH)~~ !

$\because k+1 \geq 2$

Since $k+1 \geq 2$

(In fact, $k+1 \geq 5$)

$\therefore (k+1)! > 2^{k+1}$

POMI "Take 2"

A sequence $P(1), \cdots$ of statement are all true if

(i) $P(1) \wedge P(2)$ is true.

(ii) For $k \in \mathbb{N}$, if $P(k)$ & $P(k+1)$ are true, then $P(k+2)$ is true.

6. Let $a_1 = 2$, $a_2 = 3$, & $a_{n+2} = 3a_{n+1} - 2a_n$ $\forall n \in \mathbb{N}$

prove: $a_n = 2^{n-1} + 1$ $(P(n))$

$\forall n \in \mathbb{N}$

Base cases: For $n=1$

$a_1 = 2 = 2^{1-1} + 1$ ✓

For $n=2$

$a_2 = 3 = 2^{2-1} + 1$ ✓

So $P(1)$ & $P(2)$ hold.

IH: Let $k \in \mathbb{N}$. Assume $P(k)$ & $P(k+1)$ hold.

I.E. $a_k = 2^{k-1} + 1$ & $a_{k+1} = 2^k + 1$ ... (IH)

Ind Conclusion: we want to show $P(k+2)$ holds, I.E.

$$a_{k+2} = 2^{k+1} + 1$$

Now,

$$a_{k+2} = 3a_{k+1} - 2a_k \quad \text{by definition}$$

$$= 3(2^k+1) - 2(2^{k-1}+1) \quad \text{By (IH)}$$

$$= 3 \times 2^k + 3 - 2 \times 2^{k-1} - 2$$

$$= 3 \times 2^k - 2 \cdot 2^{k-1} + 1$$

$$= 2^k(3-1) + 1$$

$$\boxed{2 \cdot 2^{k-1} = 2^k}$$

$$= 2^k \cdot 2 + 1$$

$$= 2^{k+1} + 1$$