Oct 2nd, 15

$f(\mathbb{Z}) = \{y \in \mathbb{Z} : f(x) = y \text{ For some } x \in \mathbb{Z}\}$
$= \{f(x) : x \in \mathbb{Z}\}$

$f : \mathbb{Z} \to \mathbb{Z}$

$$f(n) = \max\{1001, n\} = \begin{cases} 1001 & n \leq 1001 \\ n & n \geq 1001 \end{cases}$$

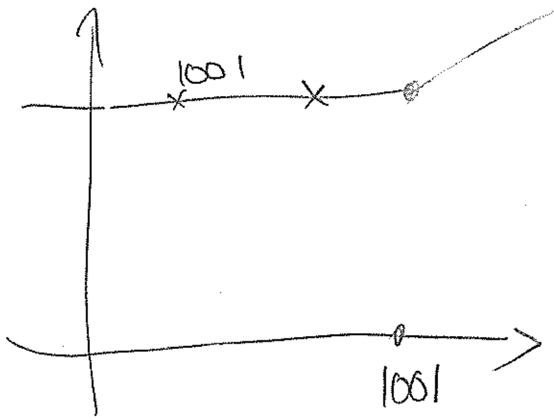
is f injective?
is f surjective?

$\forall m, n \in \mathbb{Z}, m \neq n \Rightarrow f(m) \neq f(n)$

$\equiv$

$\forall m, n \in \mathbb{Z}, f(m) = f(n) \Rightarrow m = n$

No!
$f(1) = f(2) = 1001$,
$1 \neq 2$, $1, 2 \in \mathbb{Z}$



$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$, s.t $f(x) = y$

NO! E.g. $1000 \in \mathbb{Z}$
But $f(n) \neq 1000$ For any $n \in \mathbb{Z}$

Let $n \in \mathbb{Z}$, either $n \geq 1001$ OR $n \leq 1001$, If $n \leq 1001$, then $f(n) = 1001 \neq 1000$.
If $n \geq 1001$, Then $f(n) = n \geq 1001 > 1000$.

E.g. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f(-1) = f(1) = 1 \quad \because -1 \neq 1, \quad -1, 1 \in \mathbb{R}$.

not injective

not surjective: $-1 \neq f(x) = x^2$ for any $x \in \mathbb{R}$.

$x^2 \geq 0 > -1$

continued: Let $x_1, x_2 \in \mathbb{R}_{\geq 0}$. suppose $f(x_1) = f(x_2)$

so $x_1^2 = x_2^2$

$x_1^2 - x_2^2 = 0$

$(x_1 + x_2)(x_1 - x_2) = 0$

$x_1 + x_2 = 0$ OR $x_1 - x_2 = 0$

$x_1 = -x_2$ OR $x_1 = x_2$

$\because$ $x_2 \geq 0$, $x_1 < 0$, not part of domain

$\therefore$ $x_1 = x_2$.

3. Show that if $r \in \mathbb{R} - \mathbb{Q}$ then $\frac{1}{r} \in \mathbb{R} - \mathbb{Q}$

prove by contrapositive

$\frac{1}{r} \in \mathbb{Q} \Rightarrow r \in \mathbb{Q}$

Suppose $\frac{1}{r}$ is rational.

Then $\exists\ a, b \in \mathbb{Z}, a, b \neq 0$ such that $\frac{1}{r} = \frac{a}{b}$.

Then $\frac{r}{1} = \frac{b}{a} \in \mathbb{Q}$

Let $x \in \mathbb{R}$, show that $x^2 - x > 0 \iff x \notin [0,1]$.

2. $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$ $\quad$ $P \iff Q \equiv \neg P \iff \neg Q$

$Q \Rightarrow P \equiv \neg P \Rightarrow \neg Q$

$x^2 - x \leq 0 \iff x \in [0,1]$

5. Contrapositive:

IF $a \mid c$ then either $a \mid b$ OR $a \nmid b+c$.

$$\equiv P \Rightarrow (Q \cup R)$$
$$(P \wedge \neg Q) \Rightarrow R$$

suppose $a \mid c$, suppose also $a \mid b+c$

We want to conclude that $a \mid b$.

Since $a \mid c$. and $a \mid b+c$, $a \mid (b+c-c) = b$

By divisibility of integer combination.

$$P \Rightarrow (Q \wedge R) \equiv (P \wedge \neg Q) \Rightarrow R$$

5. prove by contradiction:

suppose $a \nmid b$ and $a \mid b+c$

Want: $a \nmid c$

assume towards contradiction, $a \mid c$.

$a \mid b+c$ & $a \mid c$, so $a \mid b+c-c = b$

contradiction.

6. Show that the sum of the first n odd positive integer equals n^2.

$1+2+3 \cdots +100$

$1+3+5 \cdots +2n-1$

$2n-1+2n-3+2n-5 \cdots +1$

$2n \quad 2n \quad 2n \cdots \quad 2n$

$\underbrace{2n \cdot n}_{\text{to n times}} = 2n^2$ divides by 2 n^2.

Hilroy

sum & product notation

$$\sum_{1\le i\le n} = \sum_{i=1}^{n} x_i = x_1 + x_2 \dots + x_n$$

$$\sum_{S} = \text{Sum of elements of } S$$

$$\sum_{x\in\emptyset} = 0 \text{ ; convention} \qquad \prod_{x\in\emptyset} x = 1$$

$$\sum_{j=k}^{2k} \frac{1}{j} = \frac{1}{k} + \frac{1}{k+1} + \dots \frac{1}{2k}$$

$$\sum_{j=2} j = 0 \rightarrow \prod_{j=2} j = 1$$

$$\prod_{\substack{\text{prime}\\ p\le 3}} \left(1-\frac{1}{p^2}\right) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)$$

Factorial :

$$n! = \prod_{j=1}^{n} j \qquad n\in\mathbb{N}\cup\{0\}$$

$0! = 1$

$1! = 1$ $\qquad (n+1)! = (n+1)\,n!$

$2! = 2\times 1$

$3! = 3\times 2\times 1$

Challenge: Find $n\in\mathbb{N}$ s.t. $(1!+2! \dots + n!) \mid (n+1)!$