41P1

Q1. I enjoy trying to discover and write MATH 135 proofs.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q2.When I have difficulties with MATH 135 proofs, I know I can handle them.

A) Strongly disagree

B) Disagree

C) Neither agree nor disagree

D) Agree

E) Strongly agree

Q3. Let $n \in \mathbb{Z}$. Consider the following implication.

$$\text{If } \overbrace{(\forall x \in \mathbb{R},\ x \leq 0 \ \vee \ x+1 > n)}^{H}, \text{ then } \underbrace{n = 1}_{C}.$$

The contrapositive of this implication is

~~A)~~ If $n = 1$, then $(\forall x \in \mathbb{R},\ x \leq 0 \ \vee \ x+1 > n)$.

~~B)~~ If $n = 1$, then $(\exists x \in \mathbb{R},\ x > 0 \ \wedge \ x+1 \leq n)$.

~~C)~~ If $n \neq 1$, then $(\exists x \in \mathbb{R},\ x \geq 0 \ \wedge \ x+1 < n)$.

~~D)~~ If $n \neq 1$, then $(\forall x \in \mathbb{R},\ x \leq 0 \ \vee \ x+1 > n)$.

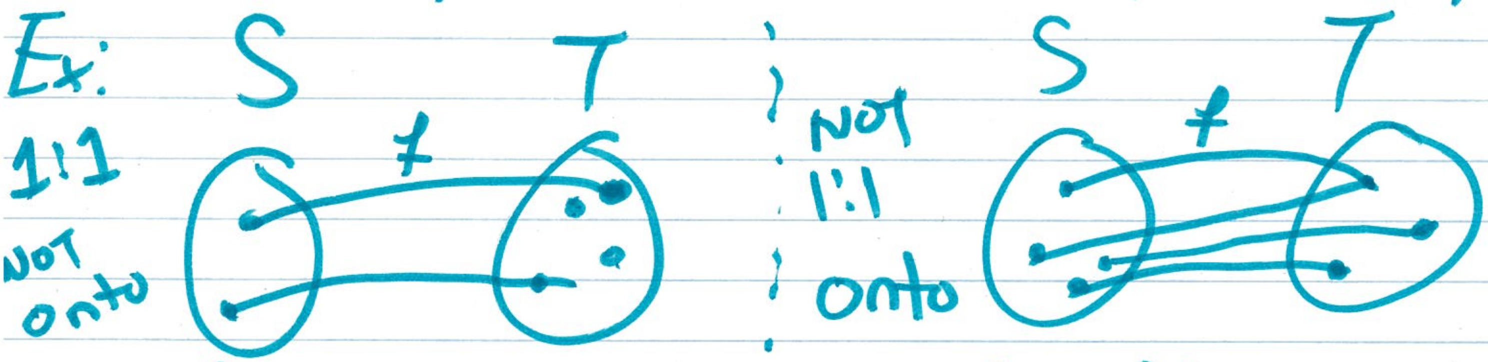
$\boxed{\text{E)}}$ None of the above.

Injections & Surjections.

Def'n: Let S & T be sets. A function $f: S \to T$ is said to be

(i) Injective (or one to one or 1:1) iff $\forall x, y \in S$ $f(x) = f(y) \Rightarrow x = y$.

Ex:



(ii) Surjective (or onto) iff $\forall y \in T$ $\exists x \in S$ s.t. $f(x) = y$.

Ex: Prove $f: \mathbb{R} \xrightarrow{\text{to}} \mathbb{R}$ is not injective, $f(x) = x^2$

$x \mapsto x^2$

maps to

Pf: Note that

$f(-1) = (-1)^2 = 1 = (1)^2 = f(1)$ BUT

$-1 \neq 1$. Thus, f is not 1:1. $\blacksquare$.

Ex: Prove that $f: \mathbb{R} \to \mathbb{R}$ is 1:1.
$x \mapsto 2x^3+1$

Pf: Let $x, y \in \mathbb{R}$ s.t. $f(x) = f(y)$. Then

$$2x^3+1 = 2y^3+1$$
$$x^3 = y^3$$
$$\sqrt[3]{x^3} = \sqrt[3]{y^3}$$

$x = y$. Thus, f is injective ∎.

Ex: Prove that $f: \mathbb{R} \to (-\infty, 1)$ is onto
$x \mapsto 1-e^{-x}$

Need to show every $y \in (-\infty, 1)$ has some $x \in \mathbb{R}$ with $f(x) = y$.

Pf: Take $x = -\ln(1-y)$ for any $y \in (-\infty, 1)$.
Then $f(x) = 1-e^{-x} = 1-e^{-(-\ln(1-y))}$

$$= 1-e^{\ln(1-y)} = 1-(1-y)$$
$$= y.$$

$\therefore f$ is onto ∎.

Uniqueness: $\exists!$ "There exists a unique."

To prove uniqueness, either

(i) Assume $\exists\, x, y \in S$ s.t. $P(x) \wedge P(y)$ is true and show $x = y$.
statement

(ii) Show $\exists\, x \in S$ s.t. $P(x)$ is true. Then use contradiction to show that if $\exists\, x, y \in S$ distinct s.t. $P(x) \wedge P(y)$ is true, then derive a contradiction.

Ex: Suppose $x \in \mathbb{R} - \mathbb{Z}$ and $m \in \mathbb{Z}$ s.t. $x < m < x+1$. Show m is unique.

Pf: Assume towards a contradiction that $\exists\, m, n \in \mathbb{Z}$ distinct s.t.

$$x < m < x+1 \text{ and } x < n < x+1.$$

Now, $0 < m - n < 1$ BUT $m - n \in \mathbb{Z}$. #. Thus, m is unique. ■.

Division Algorithm (Grade school division).

$$\overset{a}{51} = \overset{b}{7}(\overset{q}{7}) + \overset{r}{2}$$

$$-35 = 6(-6) + 1$$

"$\frac{a}{b} =$

Thm: Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then $\exists!$ $q, r \in \mathbb{Z}$ s.t. $a = bq + r$ where $0 \leq r < b$.

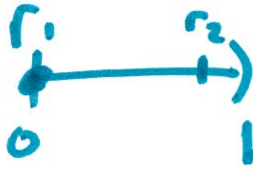
Pf: Existence: Use Well Ordering Principle on $S = \{a - bq : a - bq \geq 0 \wedge q \in \mathbb{Z}\}$

Division Algorithm Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Then $\exists!\, q, r \in \mathbb{Z}$ such that $a = qb + r$ where $0 \leq r < b$.

Proof of the division algorithm (UNIQUENESS):
Suppose that $a = q_1 b + r_1$ with $0 \leq r_1 < b$. Also, suppose that $a = q_2 b + r_2$ with $0 \leq r_2 < b$ and $r_1 \neq r_2$. Without loss of generality, we can assume $r_1 < r_2$. WLOG

(if $r_1 \neq r_2$ then one is bigger!)

Then $0 < r_2 - r_1 < b$ and $(q_1 - q_2)b = r_2 - r_1$. (Take difference of a's).



biggest gap.

By def'n

Hence $b \mid (r_2 - r_1)$. By Bounds By Divisibility, $b \leq r_2 - r_1$ which contradicts the fact that $r_2 - r_1 < b$. # no $|b|$ $\because b \in \mathbb{N}$.

Therefore, the assumption that $r_1 \neq r_2$ is false and in fact $r_1 = r_2$. But then $(q_1 - q_2)b = \underbrace{r_2 - r_1}_{=0.}$ implies $q_1 = q_2$.