Announcements

1. Office hours Thursday moved to 12 - 1:30.
2. Away Friday - Tuesday. Office hours covered by Shane Bauman (M: 9:30 - 10:30 & Tu 2 - 3:30).
3. A3 posted. Make sure by Thursday you have the link!
4. Clicker Thursday!

Prove that if $x \in \mathbb{R}$ is such that $x^3 + 7x^2 < 9$, then $x < 1.1$.

Pf: We prove the contrapositive.

Suppose $x \geq 1.1 \geq 1$. Then since $x > 0$,

$$x^3 + 7x^2 \geq (1.1)^3 + 7(1.1)^2$$

$$= \left(\frac{11}{10}\right)^3 + 7\left(\frac{11}{10}\right)^2$$

$$= \frac{1331}{1000} + 7\left(\frac{121}{100}\right)$$

$$= \frac{1331}{1000} + \frac{8470}{1000}$$

$$= \frac{9801}{1000} \geq 9. \quad \square$$

PASCAL'S Δ.

1
1 1
1 2 1
1 3 3 1

Types of Implications.

Let A, B, C be statements.

1. $A \wedge B \Rightarrow C$ (Seen: DIC. Trans. BBD)
2. $A \Rightarrow B \wedge C$

Ex: Let $S, T, \mathcal{U}$ be sets. If $(S \cup T) \subseteq \mathcal{U}$ then $S \subseteq \mathcal{U}$ and $T \subseteq \mathcal{U}$.

Pf: Suppose $S \cup T \subseteq \mathcal{U}$. If $x \in S$, Then $x \in S \cup T \subseteq \mathcal{U}$ so $x \in \mathcal{U}$. Thus $S \subseteq$
By symmetry (similarly) $T \subseteq \mathcal{U}$. ◻.

3. $A \vee B \Rightarrow C$

Ex:* $x = 1 \vee y = 2 \Rightarrow x^2y + y - 2x^2 + 4x - 2xy = 2$.

PF: If $x = 1$, then LHS $= y + y - 2 + 4 - 2y = 2 =$ RHS.
If $y = 2$, then LHS $= 2x^2 + 2 - 2x^2 + 4x - 4x = 2 =$ RH.
◻.

4. $A \Rightarrow B \vee C$. (Elimination).

Ex: If $x^2 - 7x + 12 \geq 0$ then $x \leq 3 \vee x \geq 4$

Pf: Suppose $x^2 - 7x + 12 \geq 0$ and $x > 3$.

Then, $0 \leq x^2 - 7x + 12 = \underbrace{(x-3)}_{+}\underbrace{(x-4)}_{\therefore\, \geq 0}$.

$\therefore$ $x - 4 \geq 0$ hence $x \geq 4$. $\blacksquare$.

40P4

How many years has it been since the Toronto Maple Leafs have won the Stanley Cup?

A) ~~-3~~

B) 48

C) ~~1000000~~

D) ~~1500~~

Proof By Contradiction.

Let S be a statement. Then $S \wedge \neg S$ is false.

Ex: There is no largest integer.

Pf: Assume towards a contradiction that M_0 is the largest integer. Then since $M_0 < M_0 + 1$ and $M_0 + 1 \in \mathbb{Z}$, we have contradicted the def'n of M_0. #
Thus, no largest integer exists. ⊠

Well Ordering Principle: (Axiom)

Every subset of the natural numbers that is nonempty contains a least element.

Example: Let $n \in \mathbb{Z}$ such that n^2 is even. Show that n is even.

Direct Proof: As n^2 is even, there exists a $k \in \mathbb{Z}$ such that

$$n \cdot n = n^2 = 2k.$$

Since the product of two integers is even if and only if at least one of the integers is even, we conclude that n is even.

Proof By Contradiction: Suppose that n^2 is even. Assume towards a contradiction that n is odd. Then there exists a $k \in \mathbb{Z}$ such that $n = 2k + 1$. Now,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence, n^2 is odd, a contradiction since we assumed in the statement that n^2 is even. Thus n is even.

Example: Prove that $\sqrt{2}$ is irrational.

Proof: Assume towards a contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ with $a, b \neq 0$ and $a, b \in \mathbb{N}$ (Think: Why is it okay to use $\mathbb{N}$ instead of $\mathbb{Z}$?).

Proof 1 (Well Ordering Principle): Let

$$S = \{n \in \mathbb{N} : n\sqrt{2} \in \mathbb{N}\}.$$

Since $b \in S$, we have that S is nonempty. By the Well Ordering Principle, there must be a least element of S, say k. Now, notice that

$$k(\sqrt{2} - 1) = k\sqrt{2} - k \in \mathbb{N}$$

(positive since $\sqrt{2} > \sqrt{1} = 1$). Further,

$$k(\sqrt{2} - 1)\sqrt{2} = 2k - k\sqrt{2} \in \mathbb{N}$$

and so $k(\sqrt{2} - 1) \in S$. However, $k(\sqrt{2} - 1) < k$ which contradicts the definition of k. Thus, $\sqrt{2}$ is not rational.

Proof 2 (Infinite Descent): Isolating from $\sqrt{2} = \frac{a}{b}$, we see that $2b^2 = a^2$. Thus a^2 is even hence a is even. Write $a = 2k$ for some integer k. Then $2b^2 = a^2 = (2k)^2 = 4k^2$. Hence $b^2 = 2k^2$ and so b is even. Write $b = 2\ell$ for some integer ℓ. Then repeating the same argument shows that k is even. So $a = 2k = 4m$ for some integer m. Since we can repeat this argument indefinitely and no integer has infinitely many factors of 2, we will (eventually) reach a contradiction. Thus, $\sqrt{2}$ is not rational.

Proof 3 (Simplified proof 2): Assume further that a and b share no common factor (otherwise simplify the fraction first). Then $2b^2 = a^2$. Hence a is even. Write $a = 2k$ for some integer k. Then $2b^2 = a^2 = (2k)^2 = 4k^2$ and canceling a 2 shows that $b^2 = 2k^2$. Thus b^2 is even and hence b is even. However, then a and b share a common factor, a contradiction.