Lecture 6

Note: $\{\}$ and $\emptyset$ are the empty set, a set with no elements.

Note: $\{\emptyset\}$ is NOT the empty set. It is a set with one element, the element that is the empty set.

Example: In set notation, write the set of positive integers less than 1000 and which are multiples of 7.

Instructor's Comments: Might be good to give students a minute to try this

Solution: $\{n \in \mathbb{N} : n < 1000 \wedge 7 \mid n\}$. Another answer is given by

$$\{7k : k \in \mathbb{N} \wedge k \leq 142\}$$

Note: The : symbol means "such that". Sometimes | is used as well (though because we use it for divisibility, we won't use it in this context very often if at all).

Instructor's Comments: This is the 7 minute mark

Handout or Document Camera or Class Exercise

Describe the following sets using set-builder notation:

(i) Set of even numbers between 5 and 14 (inclusive).

(ii) All odd perfect squares.

(iii) Sets of three integers which are the side lengths of a (non-trivial) triangle.

(iv) All points on a circle of radius 8 centred at the origin.

Instructor's Comments: 5 minutes to try on their own and 5 to take up

Solution:

(i) $\{6, 8, 10, 12, 14\}$ or $\{n \in \mathbb{N} : 5 \leq n \leq 14 \wedge 2 \mid n\}$

(ii) $\{(2k+1)^2 : k \in \mathbb{Z}\}$ (or $\mathbb{N}$ overlap doesn't matter!)

(iii) $\{(a, b, c) : a, b, c \in \mathbb{N} \wedge a < b + c \wedge b < a + c \wedge c < a + b\}$

(iv) $\{(x, y) : x, y \in \mathbb{R} \wedge x^2 + y^2 = 8^2\}$

Instructor's Comments: This is the 17 minute mark

Set Operations. Let S and T be sets. Define

(i) $\#S$ or $|S|$. Size of the set S.

(ii) $S \cup T = \{x : x \in S \vee x \in T\}$ (Union)

(iii) $S \cap T = \{x : x \in S \wedge x \in T\}$ (Intersection)

(iv) $S - T = \{x \in S : x \notin T\}$ (Set difference)

(v) $\bar{S}$ or S^c (with respect to universe U) the complement of S, that is

$$S^c = \{x \in U : x \notin S\} = U - S$$

(vi) $S \times T = \{(x, y) : x \in S \wedge y \in T\}$ (Cartesian Product)

Example: $(1, 2) \in \mathbb{Z} \times \mathbb{Z}$, $(2, 1) \in \mathbb{Z} \times \mathbb{Z}$, BUT $(1, 2) \neq (2, 1)$.

Note: $\mathbb{Z} \times \mathbb{Z}$ and $\{(n, n) : n \in \mathbb{Z}\}$ are different sets!!!

Example:

$$\begin{aligned}\mathbb{Z} &= \{m \in \mathbb{Z} : 2 \mid m\} \cup \{2k + 1 : k \in \mathbb{Z}\} \\ \emptyset &= \{m \in \mathbb{Z} : 2 \mid m\} \cap \{2k + 1 : k \in \mathbb{Z}\}\end{aligned}$$

Instructor's Comments: This is the 30-33 minute mark

Definition: Let S and T be sets. Then

(i) $S \subseteq T$: S is a subset of T. Every element of S is an element of T.

(ii) $S \subsetneq T$: S is a proper/strict subset of T. Every element of S is an element of T and some element of T is not in S.

(iii) $S \supseteq T$: S contains T. Every element of T is an element of S.

(iv) $S \supsetneq T$: S properly/strictly contains T. Every element of T is an element of S and some element of S is not in T.

Definition: $S = T$ means $S \subseteq T$ and $T \subseteq S$.

Example: $\{1, 2\} = \{2, 1\}$

Example: Prove $\{n \in \mathbb{N} : 4 \mid (n + 1)\} \subseteq \{2k + 1 : k \in \mathbb{Z}\}$

Proof: Let $m \in \{n \in \mathbb{N} : 4 \mid (n + 1)\}$. Then $4 \mid (m + 1)$. Thus, $\exists \ell \in \mathbb{Z}$ such that $4\ell = m + 1$. Now

$$m = 2(2\ell) - 1 = 2(2\ell) - 2 + 2 - 1 = 2(2\ell - 1) + 1.$$

Hence $m \in \{2k + 1 : k \in \mathbb{Z}\}$. ■

Instructor's Comments: This is the 40-43 minute mark. You might run out of time in the next example. Carry forward to Lecture 7 as need be.

Example: Show $S = T$ if and only if $S \cap T = S \cup T$.

Proof: Suppose $S = T$. To show $S \cap T = S \cup T$ we need to show that $S \cap T \subseteq S \cup T$ and that $S \cap T \supseteq S \cup T$

First suppose that $x \in S \cap T$. Then $x \in S$ and $x \in T$. Hence $x \in S \cup T$.

Next, suppose that $x \in S \cup T$. Then $x \in S$ or $x \in T$. Since $S = T$ we have in either case that $x \in S$ and $x \in T$. Thus $x \in S \cap T$. This shows that $S \cap T = S \cup T$ and completes the forward direction.

Now assume that $S \cap T = S \cup T$. We want to show that $S = T$ which we do by showing that $S \subseteq T$ and $T \subseteq S$.

First, suppose that $x \in S$. Then $x \in S \cup T = S \cap T$. Hence $x \in T$.

Next, suppose that $x \in T$. Then $x \in S \cup T = S \cap T$. Hence $x \in S$. Therefore, $S = T$.
$\blacksquare$

Instructor's Comments: The last two points give a good learning moment to explain when the word 'similarly' can be used. This is the 50 minute mark.