Lecture 12

Handout or Document Camera or Class Exercise

Let $n \in \mathbb{Z}$. Consider the following implication.

$$\text{If } (\forall x \in \mathbb{R},\ x \le 0 \ \vee \ x+1 > n), \text{ then } n = 1.$$

The contrapositive of this implication is

A) If $n = 1$, then $(\forall x \in \mathbb{R},\ x \le 0 \ \vee \ x+1 > n)$.

B) If $n = 1$, then $(\exists x \in \mathbb{R},\ x > 0 \ \wedge \ x+1 \le n)$.

C) If $n \neq 1$, then $(\exists x \in \mathbb{R},\ x \ge 0 \ \wedge \ x+1 < n)$.

D) If $n \neq 1$, then $(\forall x \in \mathbb{R},\ x \le 0 \ \vee \ x+1 > n)$.

E) None of the above.

Solution: None of the above (Watch the inequality signs above!).

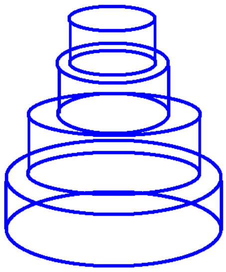
Instructor's Comments: This is the 5 minute mark. You will likely want to repoll the students (when I first gave this problem, many of my students got this wrong).

Instructor's Comments: This is a catch up lecture where if anything form the previous lectures took too long, then you can use this lecture to catch up. The only thing I would do in this lecture is show them sigma notation which I will do first and then give the class a lot of time to do practice problems.

Introduction to Summations

Example: Tower of Hanoi:

In this modified version of the Tower of Hanoi, we create a tower with levels where each level is a cylinder of height 1 and increasing radius begining with 1 and increasing by 1 at each level. Below is a level 4 Tower of Hanoi



Question: What is the volume of the 4 level Tower of Hanoi?

Solution:

$$\begin{aligned} V_{Tower} &= V_1 + V_2 + V_3 + V_4 \\ &= \pi(1)^2(1) + \pi(2)^2(1) + \pi(3)^2(1) + \pi(4)^2(1) \\ &= \pi + 4\pi + 9\pi + 16\pi \\ &= 30\pi \end{aligned}$$

Question: What about computing the volume of the 100 level Tower of Hanoi?

Solution:

$$\begin{aligned} V_{Tower} &= V_1 + V_2 + ... + V_{100} \\ &= \pi(1)^2(1) + \pi(2)^2(1) + ... + \pi(100)^2(1) \\ &= \pi + 4\pi + ... + 10000\pi \end{aligned}$$

Note: There are two concerns here. How do we evaluate this last sum? How to we write the above sum nicely and more formally without using dots?

Instructor's Comments: This is the 10 minute mark.

Sigma and Pi Notation

Definition: Let $\{a_1, ..., a_n\}$ be a sequence of n real numbers. We write

$$\sum_{i=1}^{n} a_i := a_1 + a_2 + ... + a_n.$$

We call i the index variable, 1 is the starting number, n is the upper bound. We can also write

$$\sum_{x \in S} x$$

to mean the sum of elements in S.

Instructor's Comments: Make sure you discuss the := symbol.

Similarly, we define

$$\prod_{i=1}^{n} a_i := a_1 a_2 ... a_n \qquad \prod_{x\in S} = \text{ Product of elements in S}$$

We make the following conventions when $j > k$ are integers (that is, the start index exceeds the end index)

$$\sum_{i=j}^{k} a_i = \sum_{x\in\emptyset} = 0$$

and further,

$$\prod_{i=j}^{k} a_i = \prod_{x\in\emptyset} = 1$$

Note: Sums are linear:

$$\text{For } c, j, k \in \mathbb{Z}, \sum_{i=j}^{k}(ca_i \pm b_i) = c\sum_{i=j}^{k} a_i \pm \sum_{i=j}^{k} b_i$$

Example:

(i) $\displaystyle\sum_{i=1}^{4} i^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 = 1 + 4 + 9 + 16 = 30$

(ii) $\displaystyle\prod_{i=1}^{4} i^2 = (1)^2(2)^2(3)^2(4)^2 = (1)(4)(9)(16) = 576$

(iii) $\displaystyle\sum_{i=1}^{3.5} i = 1 + 2 + 3 = 6$

(iv) For $k \in \mathbb{N}$ fixed, $\displaystyle\sum_{i=k}^{2k} 1/i = 1/k + 1/(k+1) + ... + 1/(2k)$.

(v) So we can write the 100 level tower of Hanoi volume as $\displaystyle\sum_{i=1}^{100} \pi i^2 = \pi \sum_{i=1}^{100} i^2$

Definition: We define the factorial notation for $n \geq 0$ an integer by $n! := \displaystyle\prod_{i=1}^{n} i$. Note $0! = 1$.

Example: $4! = (4)(3)(2)(1) = 24$.

Note: We will see next week how to compute the sum in our volume computation of the 100 level Tower of Hanoi.

Instructor's Comments: This is the 25-30 minute mark

Handout or Document Camera or Class Exercise

Instructor's Comments: I suggest letting students work on these and either getting them to write solutions on the board or at least telling you which ones they want to see solved.

Try some of the following problems:

- $\min\{a, b\} \leq \frac{a+b}{2}$ for all real numbers a and b.
- Let x be real. Then $x^2 - x > 0$ if and only if $x \notin [0, 1]$.
- If r is irrational, then $\frac{1}{r}$ is irrational.
- There do not exist integers p and q satisfying $p^2 - q^2 = 10$.
- The complete real solution to $x^2 + y^2 - 2y = -1$ is $(x, y) = (0, 1)$.
- Let S and T be sets with respect to a universe U. Prove that $\overline{S \cap T} \subseteq \overline{S} \cup \overline{T}$.
- Let $a, b, c \in \mathbb{Z}$. Prove that if $a \nmid b$ and $a \mid (b + c)$, then $a \nmid c$.

Instructor's Comments: Hints in order

(i) **Direct proof with cases**

(ii) **iff, contrapositive**

(iii) **contrapositive**

(iv) **contradiction**

(v) **factor**

(vi) **set inclusion**

(vii) **contrapositive and elimination.**

Solution:

- Let $a, b \in \mathbb{R}$. Without loss of generality, suppose that $a \leq b$. Then $2\min\{a, b\} = 2a < a + b$. Hence $\min\{a, b\} \leq \frac{a+b}{2}$

- Let x be real. Then

$$\begin{aligned} x^2 - x > 0 &\Leftrightarrow x(x-1) > 0 \\ &\Leftrightarrow x > 0 \wedge x - 1 > 0 \text{ or } x < 0 \wedge x - 1 < 0 \\ &\Leftrightarrow x > 1 \text{ or } x < 0 \\ &\Leftrightarrow x \notin [0, 1]. \end{aligned}$$

- We proceed by the contrapositive. If $\frac{1}{r}$ is rational, say $\frac{1}{r} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$, then $r = \frac{b}{a} \in \mathbb{Q}$.

- Assume towards a contradiction that there exists integers p and q satisfying $p^2 - q^2 = 10$. Without loss of generality, we may assume that $p, q \geq 0$ since $p^2 = (-p)^2$ so if (p, q) is a solution, then all of $(\pm p, \pm q)$ are solutions. Factoring gives $(p-q)(p+q) = 10$. Since $p + q > 0$, we have that $p - q > 0$. Since $p - q < p + q$, we see that $p - q = 1$ and $p + q = 10$ or $p - q = 2$ and $p + q = 5$. Adding the two equalities gives $2p = 11$ and $2p = 7$, both of which are a contradiction since p is an integer.

  Instructor's Comments: The previous problem can also be solved by a parity argument.

- Isolating and factoring gives $x^2 + (y-1)^2 = 0$. Hence $x = 0$ and $y = 1$.

- Suppose that $x \in \overline{S \cap T}$. We are required to show that $x \in \overline{S} \cup \overline{T}$. By definition, $x \in U - (S \cap T)$ and hence $x \in U$ and $x \notin S \cap T$. Thus, if $x \in T$, then $x \notin S$ and so $x \in \overline{S}$. Otherwise, $x \notin T$ and hence $x \in \overline{T}$. Thus, $x \in \overline{S} \cup \overline{T}$.

- We prove the contrapositive. Suppose that $a \mid c$. Then we need to show that $a \mid b$ or $a \nmid (b+c)$. By elimination, we may assume that $a \mid (b+c)$ (otherwise $a \nmid (b+c)$ and the conclusion is true). Now, $a \mid c$ and $a \mid (b+c)$ and so by Divisibility of Integer Combinations, we have that $a \mid c(-1) + (b+c)(1)$ and hence $a \mid b$.