

## Lecture 13

### Principle of Mathematical Induction (POMI)

**Axiom:** If sequence of statements  $P(1), P(2), \dots$  satisfy

- (i)  $P(1)$  is true
- (ii) For any  $k \in \mathbb{N}$ , if  $P(k)$  is true then  $P(k + 1)$  is true

then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

**Instructor's Comments:** Here describe the domino analogy. Explain that you're creating a chain of implications  $P(1) \Rightarrow P(2), P(2) \Rightarrow P(3)$ , and so on and you want the chain to begin.

In practice, these arguments proceed as follows:

- (i) Prove the base case, that is, verify that  $P(1)$  is true
- (ii) Inductive hypothesis: Let  $k \in \mathbb{N}$  be an arbitrary number. Assume that  $P(k)$  is true.
- (iii) Inductive conclusion. Deduce that  $P(k + 1)$  is true.
- (iv) Then conclude by the Principle of Mathematical Induction (POMI) that  $P(n)$  holds

**Instructor's Comments:** Emphasize the for some part in the IH step. Note also that the induction proof needn't start at 1 (it could start at 0 or  $-1$  etc.)

**Example:** Prove that

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all  $n \in \mathbb{N}$ .

**Proof:** Let  $P(n)$  be the statement that

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

holds. We prove  $P(n)$  is true for all natural numbers  $n$  by the Principle of Mathematical Induction.

- (i) Base case: When  $n = 1$ ,  $P(1)$  is the statement that

$$\sum_{i=1}^1 i^2 = \frac{(1)((1)+1)(2(1)+1)}{6}.$$

This holds since

$$\frac{(1)((1)+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1 = \sum_{i=1}^1 i^2.$$

(ii) Inductive Hypothesis. Assume that  $P(k)$  is true for some  $k \in \mathbb{N}$ . This means that

$$\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}.$$

(iii) Inductive Step. We now need to show that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

To do this, we will start with the left hand side, reduce to the assumption made in the inductive hypothesis and then conclude the right hand side.

$$\begin{aligned} \text{LHS} &= \sum_{i=1}^{k+1} i^2 \\ &= \sum_{i=1}^k i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Inductive Hypothesis} \\ &= (k+1) \left( \frac{k(2k+1)}{6} + k+1 \right) \\ &= (k+1) \left( \frac{2k^2+k}{6} + \frac{6k+6}{6} \right) \\ &= (k+1) \left( \frac{2k^2+7k+6}{6} \right) \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \text{RHS} \end{aligned}$$

Hence,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for all natural numbers  $n$  by the Principle of Mathematical Induction. ■

**Instructor's Comments: It is important to note where you used the inductive hypothesis!**

**Note:** Now, we can finally solve the Tower of Hanoi example for the 100 level tower:

$$\begin{aligned} V_{tower} &= \sum_{i=1}^{100} V_i \\ &= \sum_{i=1}^{100} \pi i^2 (1) \\ &= \pi \sum_{i=1}^{100} i^2 \\ &= \pi \frac{(100)(101)(2(100)+1)}{6} \\ &= 338350\pi \end{aligned}$$

**Instructor's Comments: This could easily be 25-30 minutes of your lecture. The rest of the time is spent doing examples:**

Handout or Document Camera or Class Exercise

Prove that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

holds for all natural numbers  $n$ .

**Solution:**

(i) Base case:

$$\frac{(1)(1+1)}{2} = 1 = \sum_{i=1}^1 i.$$

(ii) Inductive Hypothesis. Assume that

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

holds for some  $k \in \mathbb{N}$

(iii) Inductive step. For  $k+1$ ,

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{Inductive Hypothesis} \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Therefore, the claim holds by the Principle of Mathematical Induction for all  $n \in \mathbb{N}$ .

■

**Instructor's Comments: This is the 40 minute mark**

**Instructor's Comments: An example where we don't start at 1**

**Example:** Prove that  $n! > 2^n$  for all  $n \in \mathbb{N}$  with  $n \geq 4$ .

**Proof:** We proceed by mathematical induction.

- (i) Base case: When  $n = 4$ , notice that  $4! = 24 > 16 = 2^4$  so the inequality holds in this case.
- (ii) Inductive Hypothesis: Assume that  $k! > 2^k$  for some  $k \in \mathbb{N}$  with  $k \geq 4$ .
- (iii) Inductive Step: Notice that

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)2^k && \text{Inductive Hypothesis} \\ &> (1+1)2^k && \text{Since } k \geq 4 > 1 \\ &= 2^{k+1}\end{aligned}$$

Thus, the conclusion holds for all  $k \in \mathbb{N}$  with  $k \geq 4$  by the Principle of Mathematical Induction. ■

## Handout or Document Camera or Class Exercise

Examine the following induction “proofs”. Find the mistake

**Question:** For all  $n \in \mathbb{N}$ ,  $n > n + 1$ .

**Proof:** Let  $P(n)$  be the statement:  $n > n + 1$ . Assume that  $P(k)$  is true for some integer  $k \geq 1$ . That is,  $k > k + 1$  for some integer  $k \geq 1$ . We must show that  $P(k + 1)$  is true, that is,  $k + 1 > k + 2$ . But this follows immediately by adding one to both sides of  $k > k + 1$ . Since the result is true for  $n = k + 1$ , it holds for all  $n$  by the Principle of Mathematical Induction.

**Instructor’s Comments: No base cases!**

**Question:** All horses have the same colour. (Cohen 1961).

**Proof:**

**Base Case:** If there is only one horse, there is only one colour.

**Inductive hypothesis and step:** Assume the induction hypothesis that within any set of  $n$  horses for any  $n \in \mathbb{N}$ , there is only one colour. Now look at any set of  $n + 1$  horses. Number them:  $1, 2, 3, \dots, n, n + 1$ . Consider the sets  $\{1, 2, 3, \dots, n\}$  and  $\{2, 3, 4, \dots, n + 1\}$ . Each is a set of only  $n$  horses, therefore by the induction hypothesis, there is only one colour. But the two sets overlap, so there must be only one colour among all  $n + 1$  horses.

**Instructor’s Comments: However, the logic of the inductive step is incorrect for  $n = 1$ , because the statement that “the two sets overlap” is false (there are only  $n + 1 = 2$  horses prior to either removal, and after removal the sets of one horse each do not overlap. This is the 50 minute mark**