Lecture 34

Instructor's Comments: There's a large probability that you might have extra time in this lecture - there are ways to fill that time in later lectures with some extra complex numbers proofs.

Complex Numbers

Our current view of important sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

These sets can be thought of as helping us to solve polynomial equations. However, $x^2 + 1 = 0$ has no solution in any of these sets.

Instructor's Comments: This is the 3 minute mark

Definition: A complex numbers (in standard form) is an expression of the form $x + yi$ where $x, y \in \mathbb{R}$ and i is the imaginary unit. Denote the set of complex numbers by

$$\mathbb{C} := \{x + yi : x, y \in \mathbb{R}\}$$

Example: $1 + 2i$, $3i$, $\sqrt{13} + \pi i$, 2 (or $2 + 0i$).

Note:

(i) $\mathbb{R} \subseteq \mathbb{C}$

(ii) If $z = x + yi$, then $x = \mathrm{Re}(z) = \Re(z)$ is called the real part and $y = \mathrm{Im}(z) = \Im(z)$ is called the imaginary part.

Definition: Two complex numbers $z = x + yi$ and $w = u + vi$ are equal if and only if $x = u$ and $y = v$.

Definition: A complex number $z = x + yi$ is...

(i) Purely real (or simply real) if $\Im(z) = 0$, that is, $z = x$

(ii) Purely Imaginary if $\Re(z) = 0$, that is, $x = yi$.

We turn $\mathbb{C}$ into a commutative ring by defining operations as follows:

(i) $(x + yi) \pm (u + vi) := (x \pm u) + (y \pm v)i$

(ii) $(x + yi)(u + vi) := (xu - vy) + (xv + uy)i$

By this definition, we have

$$i^2 = i \cdot i = (0 + i)(0 + i) = -1 + 0i = -1.$$

Therefore, i is a solution of $x^2 + 1$. With this in mind, you can remember multiplication just by multiplying terms as you would with polynomials before.

$$(x + yi)(u + vi) = xu + xvi + yiu + yivi = xu + (xv + yu)i + yvi^2 = xu - yv + (xv + uy)i$$

Example:

(i) $(1+2i)+(3+4i) = 4+6i$

(ii) $(1+2i)-(3+4i) = -2-2i$

(iii) $(1+2i)(3+4i) = 3-8+(4+6)i = -5+10i$

We note that $\mathbb{C}$ is a field by observing that the multiplicative inverse of a nonzero complex numbers is

$$(x+yi)^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

Exercise: If $z \in \mathbb{C}$ and $z \neq 0$, then $z \cdot z^{-1} = 1$

Instructor's Comments: This is the 20-25 minute mark.

For complex numbers u, v, w, z with v and z nonzero, the above is consistent with the usual fraction rules:

$$\frac{u}{v} + \frac{w}{z} = \frac{uz+vw}{vz} \qquad \text{and} \qquad \frac{u}{v} \cdot \frac{w}{z} = \frac{uw}{vz}$$

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$, define

$$z^0 = 1 \qquad z^1 = z \qquad z^{k+1} = z \cdot z^k$$

and further that $z^{-k} := (z^{-1})^k$. With these definitions, the usual exponent rules hold, namely

$$z^{m+n} = z^m \cdot z^n \qquad (z^m)^n = z^{mn}$$

for $m, n \in \mathbb{Z}$.

Example: Write $\frac{1+2i}{3-4i}$ in standard form.

Solution:

$$\begin{aligned}
\frac{1+2i}{3-4i} &= (1+2i)(3-4i)^{-1} \\
&= (1+2i)\left(\frac{3}{3^2+4^2} - \frac{(-4)}{3^2+4^2}i\right) \\
&= (1+2i)\left(\frac{3}{25} + \frac{4}{25}i\right) \\
&= \frac{3}{25} - \frac{8}{25} + \left(\frac{4}{25} + \frac{6}{25}\right)i \\
&= \frac{-5}{25} + \frac{10}{25}i \\
&= \frac{-1}{5} + \frac{2}{5}i
\end{aligned}$$

Instructor's Comments: This is the 30 minute mark

Handout or Document Camera or Class Exercise

Express the following in standard form

(i) $z = \frac{(1-2i)-(3+4i)}{5-6i}$

(ii) $w = i^{2015}$

Solution:

(i)

$$\begin{aligned} z &= ((1-2i)-(3+4i))(5-6i)^{-1} \\ &= (-2-6i)\left(\frac{5}{5^2+6^2} - \frac{(-6)}{5^2+6^2}i\right) \\ &= (-2-6i)\left(\frac{5}{61} + \frac{6}{61}i\right) \\ &= \frac{-10}{61} + \frac{36}{61} + \left(\frac{-12}{61} - \frac{30}{61}\right) i \\ &= \frac{26}{61} - \frac{42}{61}i \end{aligned}$$

(ii) Recall that $i^2 = -1$ and $i^4 = 1$. Thus,

$$\begin{aligned} w &= i^{2015} \\ &= (i^4)^{503} \cdot i^3 \\ &= 1^{503} \cdot i^2 \cdot i \\ &= -i \end{aligned}$$

Instructor's Comments: This is the 40 minute mark - you can easily go on to the next lecture or use this time to catch up.