

## Lecture 39

**Theorem:** Complex  $n$ th Roots Theorem (CNRT) Any nonzero complex number has exactly  $n \in \mathbb{N}$  distinct  $n$ th roots. The roots lie on a circle of radius  $|z|$  centred at the origin and spaced out evenly by angles of  $2\pi/n$ . Concretely, if  $a = re^{i\theta}$ , then solutions to  $z^n = a$  are given by  $z = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}$  for  $k \in \{0, 1, \dots, n-1\}$ .

**Proof:** The proof is like the example yesterday and is left as additional reading. ■

**Definition:** An  $n$ th root of unity is a complex number  $z$  such that  $z^n = 1$ . These are sometimes denoted by  $\zeta_n$ .

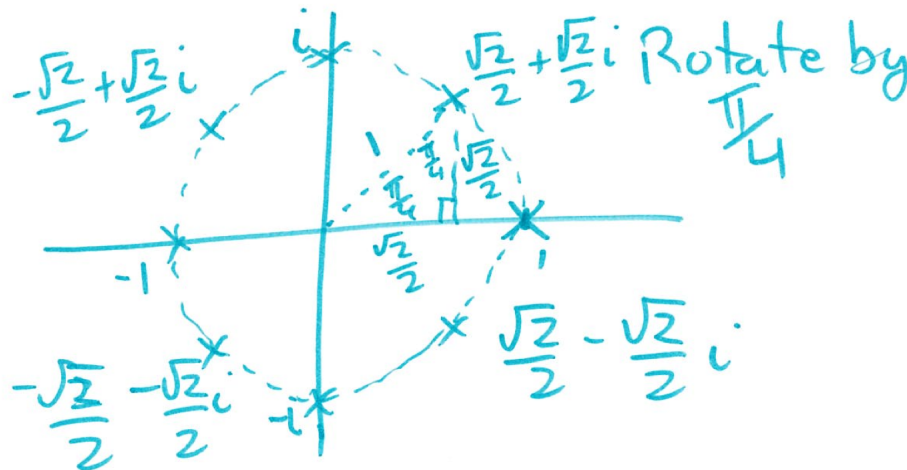
**Example:**  $-1$  is a second root of unity (and a fourth root of unity and a sixth root of unity etc.)

**Instructor's Comments:** This is the 10 minute mark; though likely the previous lecture spilled over to this lecture.

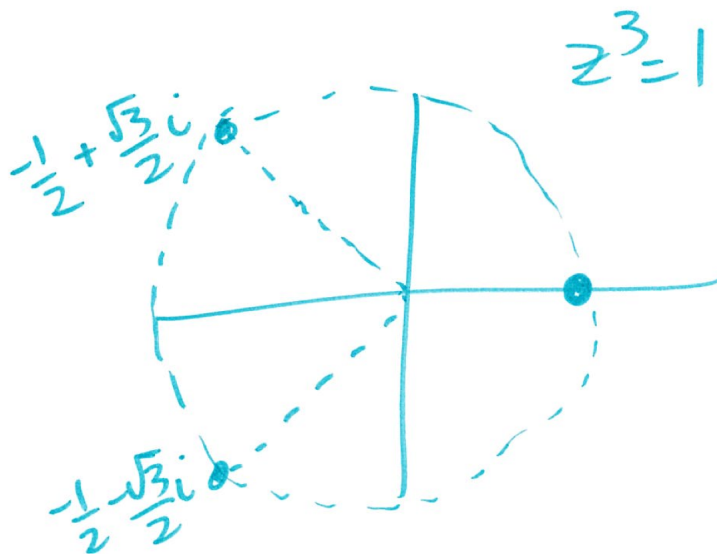
Handout or Document Camera or Class Exercise

Find all eighth roots of unity in standard form.

**Solution:** We want to solve  $z^8 = 1$ . We know that  $\{\pm 1, \pm i\}$  are solutions. We can draw to find the rest:



For another example, look at  $z^3 = 1$ :



**Example:** Solve  $z^5 = -16\bar{z}$ .

**Instructor's Comments:** Get students to guess the total number of solutions. Also get them to find a solution by inspection. The answer is surprising!

**Solution:** This is a tricky problem. One could convert to polar coordinates but I prefer to reason as follows. If I can't solve the equation as written, maybe I can simplify by taking lengths on both sides.

$$|z^5| = |z|^5 = |-16\bar{z}| = 16|\bar{z}| = 16|z|$$

This gives  $|z|^5 = 16|z|$ . Hence  $|z|^5 - 16|z| = 0$  giving  $|z|(|z|^4 - 16) = 0$ . This gives either  $|z| = 0$  which translates to  $z = 0$  or  $|z|^4 = 16$  which gives  $|z| = 2$ . So assuming that  $z \neq 0$ , we multiply the original equation by  $z$  to yield

$$z^6 = -16z\bar{z} = -16|z|^2 = -64$$

but this question we solved before! Therefore,

$$z \in \{0, \pm 2i, \pm\sqrt{3} \pm i\}$$

Thus, there are seven solutions!

**Instructor's Comments:** This is the 40 minute mark; if you spilled over from the previous lecture, this is the 50 minute mark. Otherwise do the next problem (which is one we did before)

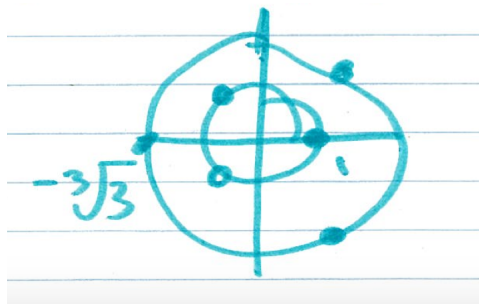
**Example:** Solve  $z^6 + 2z^3 - 3 = 0$ .

**Proof:** From before, we factored this to  $(z^3 - 1)(z^3 + 3) = 0$  and thus  $z^3 = 1$  or  $z^3 = -3$ . From CNRT, we see that the solutions to  $z^3 = 1 = \cos(0) + i \sin(0)$  are given by

$$z \in \{e^{i \cdot 0}, e^{i \cdot 2\pi/3}, e^{i \cdot 4\pi/3}\}$$

and solutions to  $z^3 = -3 = 3(\cos(\pi) + i \sin(\pi))$  are given by

$$z \in \{\sqrt[3]{3}e^{i \cdot \pi/3}, \sqrt[3]{3}e^{i \cdot \pi}, \sqrt[3]{3}e^{i \cdot 5\pi/3}\}$$



This completes the question. ■