Lecture 43

Theorem: Rational Roots Theorem (RRT) If $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{Z}[x]$ and $r = \frac{s}{t} \in \mathbb{Q}$ is a root of $f(x)$ over $\mathbb{Q}$ in lowest terms, then $s \mid a_0$ and $t \mid a_n$.

Proof: Plug r into $f(x)$:

$$0 = a_n(\tfrac{s}{t})^n + ... + a_1(\tfrac{s}{t}) + a_0.$$

Multiply by t^n

$$0 = a_n s^n + a_{n-1}s^{n-1}t + ... + a_1 s t^{n-1} + a_0 t^n.$$

Rearranging gives

$$a_0 t^n = -s(a_n s^{n-1} + a_{n-1}s^{n-2}t + ... + a_1 t^{n-1})$$

and hence $s \mid a_0 t^n$. Since $\gcd(s,t) = 1$, we see that $\gcd(s, t^n) = 1$ (following from GCDPF) and hence $s \mid a_0$ by Coprimeness and Divisibility. Similarly, $t \mid a_n$. ■

Example: Find the roots of

$$2x^3 + x^2 - 6x - 3 \in \mathbb{R}[x]$$

Solution: By the Rational Roots Theorem, if r is a root, then writing $r = \frac{s}{t}$, we have that $s \mid -3$ and $t \mid 2$. This gives the following possibilities for r:

$$\pm 1, \pm 3, \pm\frac{3}{2}, \pm\frac{1}{2}$$

Trying each of these possibilities one by one shows that $r = -\frac{1}{2}$ is a root since

$$2(\tfrac{-1}{2})^3 + (\tfrac{-1}{2})^2 - 6(\tfrac{-1}{2}) - 3 = \tfrac{-1}{4} + \tfrac{1}{4} + 3 - 3 = 0$$

Hence $(x + \frac{1}{2})$ or $(2x+1)$ is a factor. By long division (or grouping and factoring), we see that

$$2x^3 + x^2 - 6x - 3 = (2x+1)(x^2 - 3) = (2x+1)(x - \sqrt{3})(x + \sqrt{3})$$

Hence all real roots are given by $\frac{-1}{2}, \pm\sqrt{3}$. ■

Instructor's Comments: This is the 15 minute mark.

Handout or Document Camera or Class Exercise

Factor $x^3 - \frac{32}{15}x^2 + \frac{1}{5}x + \frac{2}{15}$ as a product of irreducible polynomials over $\mathbb{R}$.

Solution: The above polynomial is equal to

$$\tfrac{1}{15}(15x^3 - 32x^2 + 3x + 2) = f(x)$$

By the Rational Roots Theorem, possible roots are

$$\pm 1, \pm\tfrac{1}{3}, \pm\tfrac{1}{5}, \pm\tfrac{1}{15}, \pm 2, \pm\tfrac{2}{3}, \pm\tfrac{2}{5}, \pm\tfrac{2}{15},$$

Note that $x = 2$ is a root. Hence by the Factor Theorem, $x - 2$ is a factor. By long division:

$$
\begin{array}{r}
15x^2-2x-1 \\
x-2\,\overline{)\,15x^3-32x^2+3x+2} \\
15x^2-30x^2 \\
\hline
-2x^2+3x \\
-2x^2+4x \\
\hline
-x+2
\end{array}
$$

we have that $f(x) = \frac{1}{15}(x-2)(15x^2 - 2x - 1) = \frac{1}{15}(x-2)(5x+1)(3x-1)$ completing the question. ■

Instructor's Comments: This is the 30 minute mark

Example: Prove that $\sqrt{7}$ is irrational.

Proof: Assume towards a contradiction that $\sqrt{7} = x \in \mathbb{Q}$. Square both sides gives

$$7 = x^2 \qquad \Longrightarrow \qquad 0 = x^2 - 7$$

Therefore, as a polynomial, $x^2 - 7$ has a rational root. By the Rational Root Theorem, the only possible rational roots are given by $\pm 1, \pm 7$. By inspection, none of these are roots:

$$(\pm 1)^2 - 7 = -6 \neq 0 \qquad (\pm 7)^2 - 7 = 42 \neq 0$$

Hence, x cannot be rational. ■

Instructor's Comments: This is the 35 minute mark

Handout or Document Camera or Class Exercise

Prove that $\sqrt{5} + \sqrt{3}$ is irrational.

Solution: Assume towards a contradiction that $\sqrt{5} + \sqrt{3} = x \in \mathbb{Q}$. Squaring gives

$$5 + 2\sqrt{15} + 3 = x^2 \qquad \implies \qquad 2\sqrt{15} = x^2 - 8$$

Squaring again gives

$$60 = x^4 - 16x^2 + 64 \qquad \implies \qquad 0 = x^4 - 16x^2 + 4$$

By the Rational Roots Theorem, the only possible roots are

$$\pm 1, \pm 2, \pm 4$$

A quick check shows that none of these work. ■

Instructor's Comments: This is the 45 minute mark

Theorem: (Conjugate Roots Theorem (CJRT)) If $c \in \mathbb{C}$ is a root of a polynomial $p(x) \in \mathbb{R}[x]$ (over $\mathbb{C}$) then $\overline{c}$ is a root of $p(x)$.

Proof: Write $p(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{R}[x]$ with $p(c) = 0$. Then:

$$\begin{aligned}
p(\overline{c}) &= a_n(\overline{c})^n + ... + a_1\overline{c} + a_0 \\
&= \overline{a_n(c)^n} + ... + \overline{a_1 c} + \overline{a_0} && \text{Since coefficients are real and PCJ.} \\
&= \overline{a_n(c)^n + ... + a_1 c + a_0} && \text{By PCJ} \\
&= \overline{p(c)} \\
&= 0
\end{aligned}$$

Instructor's Comments: This is the 50 minute mark.