Lecture 15

Instructor's Comments: If you did the surveys, you could go over them at the beginning

Handout or Document Camera or Class Exercise

Fibonacci Sequence Definition: Define a sequence by $f_1 = 1$, $f_2 = 1$ and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{For all } n \geq 3$$

so $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, and so on.

(i) Prove that $\sum_{r=1}^{n} f_r^2 = f_n f_{n+1}$ for all $n \in \mathbb{N}$.

(ii) Prove that $f_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{N}$.

Solution: We prove only the first one. The second can be found on the Math 135 resources page

`http://www.cemc.uwaterloo.ca/~cbruni/Math135Resources.php`

(i) Base case: $n = 1$

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{n} f_r^2 \\ &= \sum_{r=1}^{1} f_r^2 \\ &= f_1^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

and

$$\text{RHS} = f_n f_{n+1} = f_1 f_2 = (1)(1) = 1 = \text{LHS}$$

(ii) Inductive Hypothesis. Assume that

$$\sum_{r=1}^{k} f_r^2 = f_k f_{k+1}$$

holds for some $k \in \mathbb{N}$.

(iii) Inductive Step. We want to show that

$$\sum_{r=1}^{k+1} f_r^2 = f_{k+1} f_{k+2}.$$

We begin with the left and proceed towards the right

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1} f_r^2 \\
&= \sum_{r=1}^{k} f_r^2 + f_{k+1}^2 \\
&= f_k f_{k+1} + f_{k+1}^2 && \text{Induction Hypothesis} \\
&= f_{k+1}(f_k + f_{k+1}) \\
&= f_{k+1} f_{k+2} && \text{By definition of Fibonacci Sequence} \\
&= \text{RHS}
\end{aligned}$$

Hence $\sum_{r=1}^{n} f_r^2 = f_n f_{n+1}$ for all $n \in \mathbb{N}$ by the Principle of Mathematical Induction. ■

Instructor's Comments: This easily is the 20-30 minute mark. Students might struggle with the notation.

Definition: Closed form: "Easy to put into a calculator" (This is not a formal definition!)

Example: Find a closed form expression for

$$P_n = \prod_{r=2}^{n} \left(1 - \frac{1}{r^2}\right)$$

where $n \geq 2$ and prove it is correct by induction.

Proof: We begin with some guessing and napkin (discovery) work.

$$P_2 = \prod_{r=2}^{2} \left(1 - \frac{1}{r^2}\right) = \left(1 - \frac{1}{2^2}\right) = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

$$P_3 = \prod_{r=2}^{3} \left(1 - \frac{1}{r^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) = (1 - \tfrac{1}{4})(1 - \tfrac{1}{9}) = \tfrac{3}{4} \cdot \tfrac{8}{9} = \tfrac{2}{3} = \tfrac{4}{6}$$

$$P_4 = \prod_{r=2}^{4} \left(1 - \frac{1}{r^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) = (1 - \tfrac{1}{4})(1 - \tfrac{1}{9})(1 - \tfrac{1}{16}) = \tfrac{3}{4} \cdot \tfrac{8}{9} \cdot \tfrac{15}{16} = \tfrac{5}{8}$$

Claim: $P_5 = \frac{6}{10}$ and in general $P_n = \frac{n+1}{2n}$ for all $n \geq 2$. We prove this by induction.

(i) Base case: $n = 2$

$$P_2 = \prod_{r=2}^{2} \left(1 - \frac{1}{r^2}\right) = \left(1 - \frac{1}{2^2}\right) = 1 - \tfrac{1}{4} = \tfrac{3}{4} = \tfrac{n+1}{2n}$$

(ii) Inductive Hypothesis. Assume that $P(k)$ is true for some $k \geq 2$ and $k \in \mathbb{N}$, that is, assume

$$\prod_{r=2}^{k} \left(1 - \frac{1}{r^2}\right) = \frac{k+1}{2k}$$

(iii) Inductive Step. We want to show that

$$\prod_{r=2}^{k+1} \left(1 - \frac{1}{r^2}\right) = \frac{(k+1)+1}{2(k+1)} = \frac{k+2}{2k+2}$$

We proceed starting from the left.

$$
\begin{aligned}
\text{LHS} &= \prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right) \\
&= \prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\cdot\frac{(k+1)^2-1}{(k+1)^2} && \text{Inductive Hypothesis} \\
&= \frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2} \\
&= \frac{k+1}{2k}\cdot\frac{k(k+2)}{(k+1)^2} \\
&= \frac{k+2}{2(k+1)} \\
&= \text{RHS}
\end{aligned}
$$

Therefore, by the Principle of Mathematical Induction, we have that

$$P_n = \tfrac{n+1}{2n}$$

for all $n \geq 2$. ■

Instructor's Comments: This is the 50 minute mark.