Lecture 3

Handout or Document Camera or Class Exercise

Find the flaw in the following arguments:

(i) (Last class)

(ii)

$$
\begin{aligned}
x &= \frac{\pi+3}{2} \\
2x &= \pi + 3 \\
2x(\pi-3) &= (\pi+3)(\pi-3) \\
2\pi x - 6x &= \pi^2 - 9 \\
9 - 6x &= \pi^2 - 2\pi x \\
9 - 6x + x^2 &= \pi^2 - 2\pi x + x^2 \\
(3-x)^2 &= (\pi-x)^2 \\
3 - x &= \pi - x \qquad \text{ERROR: } |3-x| = |\pi - x| \\
3 &= \pi
\end{aligned}
$$

(iii) For $x \in \mathbb{R}$,

$$
\begin{aligned}
(x-1)^2 &\geq 0 \\
x^2 - 2x + 1 &\geq 0 \\
x^2 + 1 &\geq 2x \\
x + \tfrac{1}{x} &\geq 2 \qquad \text{ERROR: Division by 0. Also flip sign if } x < 0
\end{aligned}
$$

Instructor's Comments: This is the 5 minute mark

Example: Let $x, y \in \mathbb{R}$. Prove that

$$5x^2y - 3y^2 \leq x^4 + x^2y + y^2$$

Proof: Since $0 \leq (x^2 - 2y)^2$, we have

$$\begin{aligned} 0 &\leq (x^2 - 2y)^2 \\ 0 &\leq x^4 - 4x^2y + 4y^2 \\ 5x^2y - 3y^2 &\leq x^4 - 4x^2y + 4y^2 + 5x^2y - 3y^2 \\ 5x^2y - 3y^2 &\leq x^4 + x^2y + y^2 \end{aligned}$$

Alternate proof:

$$\begin{aligned} \text{RHS} &= x^4 + x^2y + y^2 \\ &= x^4 + x^2y + y^2 + 5x^2y - 5x^2y + 3y^2 - 3y^2 \\ &= x^4 - 4x^2y + 4y^2 + 5x^2y - 3y^2 \\ &= (x^2 - 2y)^2 + 5x^2y - 3y^2 \\ &\geq 5x^2y - 3y^2 \\ &= \text{LHS} \end{aligned}$$

Note: To discover this proof. Play around with the given inequality on a napkin (rough work). Manipulate it until you reach a true statement. Then write your proof starting with the given true statement to reach the desired inequality. Notice that starting with the given inequality is NOT valid since you do not know whether or not it is true to begin with. New truth can only be derived from old truth. (Analogy: You need a solid foundation to build a house). Here is a sample of my napkin work:

$$\begin{aligned} 5x^2y - 3y^2 &\leq x^4 + x^2y + y^2 \\ 0 &\leq x^4 + x^2y + y^2 - 5x^2y + 3y^2 \\ 0 &\leq x^4 - 4x^2y + 4y^2 \\ 0 &\leq (x^2 - 2y)^2. \end{aligned}$$

The last statement is clearly true thus so long as I can reverse my steps, I have a valid proof. Note that you must write the proof starting with the true statement and deriving the new truth statements.

Instructor's Comments: This is the 20 minute mark

Throughout the remainder of this lecture, let A, B, C be statements.

Definition: $\neg A$ is NOT A.

A	$\neg A$
T	F
F	T

Note: : Truth tables can be used both as definitions of operators (as was done here) or in proofs (as will be done later). Make sure you understand the difference.

Definition: $A \wedge B$ is A and B. Further, $A \vee B$ is A or B.

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Instructor's Comments: This is the 26 minute mark

Handout or Document Camera or Class Exercise

Which of the following are true?

- π is irrational and $3 > 2$
- 10 is even and $1 = 2$
- 7 is larger than 6 or 15 is a multiple of 3
- $5 \leq 6$
- 24 is a perfect square or the vertex of parabola $x^2 + 2x + 3$ is $(1, 1)$
- 2.3 is not an integer
- 20% of 50 is not 10
- 7 is odd or 1 is positive and $2 \neq 2$

Solution: In order: True, False, True, True, False, True, False, True.
Note: For the last one above, the order of operations for logical operators (mathematically) is $\neg$, $\wedge$, $\vee$. If you change this order, the last bullet becomes false. This is not required knowledge in MATH 135 but you should make a note. Further, this is not consistent across programming languages.

Instructor's Comments: This is the 32 minute mark. It is possible to move this to the end of the lecture near the other similar handout if you want to avoid swapping back and forth from projector to notes.

Definition: The symbol $\equiv$ in logic means "logically equivalent", that is, in a truth table, the LHS and RHS are equivalent (share the same truth values for all possibilities; share the same truth values in columns, etc.). **Example:** Show that $\neg(\neg A) \equiv A$.
Proof:

A	$\neg A$	$\neg(\neg A)$
T	F	T
F	T	F

Since the first and last columns are equal, $A == \neg(\neg A)$.

Note: It is important to have a concluding statement like above. Make sure the reader knows why you know you have proven your statement.
Theorem: De Morgan's Law (DML)

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

We prove only the first. The second is left as an exercise.

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the fourth and the last columns are equal, we have that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ as required. ■

Instructor's Comments: It is worth noting that this is the first time an acronym is used. I am not certain if this acronym is in the textbook. This would be a good time to emphasize when using a theorem or a result, you should use the acronym or name.

Example: For Homework, prove that $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

Instructor's Comments: This is the 40 minute mark

Definition: Implication ($A \Rightarrow B$)

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

In $A \Rightarrow B$, we call A the *hypothesis* and B the *conclusion*.

Note: Notice that if the hypothesis is false, the implication is always evaluated as true. Similarly, if the conclusion is true, the implication is always evaluated as true.

Note: To **prove** $A \Rightarrow B$, we **assume** A is true and then show that B is true.

Note: To **use** $A \Rightarrow B$, we **prove** A is true and then use B as true.

Proposition: Let A and B be statements. Then $A \Rightarrow B \equiv \neg A \vee B$.

Proof:

A	B	$A \Rightarrow B$	$\neg A$	$\neg A \vee B$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the third and fifth columns are equal, we see that $A \Rightarrow B \equiv \neg A \vee B$. ■

Handout or Document Camera or Class Exercise

In the following, identify the hypothesis, the conclusion and state whether the statement is true or false.

- If $\sqrt{2}$ is rational then $2 < 3$
- If (1+1=2) then $5 \cdot 2 = 11$
- If C is a circle, then the area of C is πr^2
- If 5 is even then 5 is odd
- If $4 - 3 = 2$ then $1 + 1 = 3$

Solution: True, False, True, True, True.

Instructor's Comments: This is the 50 minute mark