Lecture 7

Quantified Statements

(i) For every natural number n, $2n^2 + 11n + 15$ is composite.

(ii) There is an integer k such that $6 = 3k$.

Symbolically, we write

(i) $\forall n \in \mathbb{N}$, $2n^2 + 11n + 15$ is composite.

(ii) $\exists k \in \mathbb{Z}$ such that $6 = 3k$.

We call $\forall$ and $\exists$ quantifiers, n and k variables, $\mathbb{N}$ and $\mathbb{Z}$ domains and the rest are called an open sentence (usually involving the variable(s)).

Note: $\forall x \in S\, P(x)$ means for all x in S, statment $P(x)$ holds. This is equivalent to $x \in S \Rightarrow P(x)$.

Proof: (of number 1 above) Let n be an arbitrary natural number. Then factoring gives $2n^2 + 11n + 15 = (2n + 5)(n + 3)$. Since $2n + 5 > 1$ and $n + 3 > 1$, we have $2n^2 + 11n + 15$ is composite.

Proof: (of number 2 above) Since $3 \cdot 2 = 6$, we see that $k = 2$ satisfies the given statement.

Example: $S \subseteq T \equiv \forall x \in S\, x \in T$

Instructor's Comments: This is the 7 minute mark

Handout or Document Camera or Class Exercise

Example: Prove that there is an $x \in \mathbb{R}$ such that $\frac{x^2+3x-3}{2x+3} = 1$.

Proof: When $x = 2$, note that $\frac{2^2+3(2)-3}{2(2)+3} = \frac{7}{7} = 1$. ■

Note: : The discovery of this proof is perhaps what is more interesting:

$$\frac{x^2+3x-3}{2x+3} = 1 \quad \Leftrightarrow \quad x^2+3x-3 = 2x+3 \quad \Leftrightarrow \quad x^2+x-6 = 0$$

and the last equation factors as $(x-2)(x+3) = 0$ and hence $x = 2$.

Instructor's Comments: This is the 17 minute mark

Note: : Vacuously true statements $\forall x \in \emptyset, P(x)$. Since there is no element in the empty set, we define this statement to always be true as a matter of convention.

Example: Let $a, b, c \in \mathbb{Z}$. If $\forall x \in \mathbb{Z}$, $a \mid (bx + c)$ then $a \mid (b + c)$.

Proof: Assume $\forall x \in \mathbb{Z}$, $a \mid (bx + c)$. Then, for example, when $x = 1$, we see that $a \mid (b(1) + c)$. Thus $a \mid (b + c)$.

Instructor's Comments: Note: If you're running short on time, this next example can be omitted

Example: $\exists m \in \mathbb{Z}$ such that $\frac{m-7}{2m+4} = 5$.

Proof: When $m = 3$, note that $\frac{m-7}{2m+4} = \frac{-3-7}{2(-3)+4} = \frac{-10}{-2} = 5$

Instructor's Comments: This should be the 26-30 minute mark

Handout or Document Camera or Class Exercise

Example: Show that for each $x \in \mathbb{R}$, we have that $x^2 + 4x + 7 > 0$.

Instructor's Comments: For the next two pages, you should give students say 5 minutes each (maybe more for the second handout) and then take them up as a class for 5 minutes each

Proof: Let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} x^2 + 4x + 7 &= x^2 + 4x + 4 - 4 + 7 \\ &= (x+2)^2 + 3 \\ &> 0 \end{aligned}$$

Handout or Document Camera or Class Exercise

Sometimes $\forall$ and $\exists$ are hidden! If you encounter a statement with quantifiers, take a moment to make sure you understand what the question is saying/asking.

Examples:

(i) $2n^2 + 11n + 15$ is never prime when n is a natural number.

(ii) If n is a natural number, then $2n^2 + 11n + 15$ is composite.

(iii) $\frac{m-7}{2m+4} = 5$ for some integer m.

(iv) $\frac{m-7}{2m+4} = 5$ has an integer solution.

Solution:

(i) $\forall n \in \mathbb{N}, 2n^2 + 11n + 15$ is not prime.

(ii) $\forall n \in \mathbb{N}, 2n^2 + 11n + 15$ is composite.

(iii) $\exists m \in \mathbb{Z}, \frac{m-7}{2m+4} = 5$.

(iv) $\exists m \in \mathbb{Z}, \frac{m-7}{2m+4} = 5$.

Instructor's Comments: This should be about the 46 minute mark

Note: : Domain is important!

Let $P(x)$ be the statement $x^2 = 2$ and let $S = \{\sqrt{2}, -\sqrt{2}\}$. Which of the following are true?

(i) $\exists x \in \mathbb{Z}, P(x)$

(ii) $\forall x \in \mathbb{Z}, P(x)$

(iii) $\exists x \in \mathbb{R}, P(x)$

(iv) $\forall x \in \mathbb{R}, P(x)$

(v) $\exists x \in S, P(x)$

(vi) $\forall x \in S, P(x)$

Solution:

(i) False

(ii) False

(iii) True

(iv) False

(v) True

(vi) True

Instructor's Comments: This is the end of the lecture.