Lecture 44

Handout or Document Camera or Class Exercise

How many of the following statements are true?

- Every complex cubic polynomial has a complex root.
- When $x^3 + 6x - 7$ is divided by a quadratic polynomial $ax^2 + bx + c$ in $\mathbb{R}[x]$, then the remainder has degree 1.
- If $f(x), g(x) \in \mathbb{Q}[x]$, then $f(x)g(x) \in \mathbb{Q}[x]$.
- Every non-constant polynomial in $\mathbb{Z}_5[x]$ has a root in $\mathbb{Z}_5$.

A) 0

B) 1

C) 2

D) 3

E) 4

Solution: The first statement is true by the Fundamental Theorem of Algebra. The second is false since $x - 1$ is a factor of the cubic polynomial and so there must be a quadratic factor as well. The third is true since $\mathbb{Q}[x]$ forms a ring. The last is false since say $f(x) = x(x-1)(x-2)(x-3)(x-4) + 1$ has no roots over $\mathbb{Z}_5[x]$. Hence the answer is 2.

Recall:

Theorem: (Conjugate Roots Theorem (CJRT)) If $c \in \mathbb{C}$ is a root of a polynomial $p(x) \in \mathbb{R}[x]$ (over $\mathbb{C}$) then $\overline{c}$ is a root of $p(x)$.

Note: This is not true if the coefficients are not real, for example $(x+i)^2 = x^2 + 2ix - 1$.

Example: Factor

$$f(z) = z^5 - z^4 - z^3 + z^2 - 2z + 2$$

over $\mathbb{C}$ as a product of irreducible elements of $\mathbb{C}[x]$ given that i is a root.

Proof: Note by CJRT that $\pm i$ are roots. By the Factor Theorem, we see that $(z-i)(z+i) = z^2 + 1$ is a factor. Note that $z - 1$ is also a factor since the sum of the coefficients is 0.Hence, $(z^2+1)(z-1) = z^3 - z^2 + z - 1$ is a factor. By long division,

$$\begin{array}{r l}
 & \quad z^2 - 2 \\
z^3 - z^2 + z - 1 & \overline{\big)\, z^5 - z^4 - z^3 + z^2 - 2z + 2} \\
 & \underline{-(z^5 - z^4 + z^3 - z^2)} \\
 & \qquad\qquad -2z^3 + 2z^2 - 2z + 2 \\
 & \qquad\qquad \overline{\qquad\qquad\qquad\qquad\quad 0}
\end{array}$$

we see that $f(z) = (z^3 - z^2 + z - 1)(z^2 - 2) = (z-i)(z+i)(z-1)(z-\sqrt{2})(z+\sqrt{2})$ is a full factorization. ■

Factor $f(z) = z^4 - 5z^3 + 16z^2 - 9z - 13$ over $\mathbb{C}$ into a product of irreducible polynomials given that $2 - 3i$ is a root.

Factors are (using the Factor Theorem and CJRT)

$$(z - (2 - 3i))(z - (2 + 3i)) = z^2 - 4z + 13$$

After long division,

$$f(z) = (z^2 - 4z + 13)(z^2 - z - 1)$$

By the quadratic formula on the last quadratic,

$$\begin{aligned} z &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Hence, $f(z) = (z - (2 - 3i))(z - (2 + 3i))(z - (1 + \sqrt{5})/2)(z - (1 - \sqrt{5})/2)$. $\blacksquare$

Theorem: (Real Quadratic Factors (RQF)) Let $f(x) \in \mathbb{R}[x]$. If $c \in \mathbb{C} - \mathbb{R}$ and $f(c) = 0$, then there exists a $g(x) \in \mathbb{R}[x]$ such that $g(x)$ is a real quadratic factor of $f(x)$.

Proof: Take

$$\begin{aligned} g(x) &= (x - c)(x - \overline{c}) \\ &= x^2 - (c + \overline{c})x + c\overline{c} \\ &= x^2 - 2\Re(c)x + |c|^2 \in \mathbb{R}[x] \end{aligned}$$

It suffices to show that $g(x)$ is a factor of $f(x)$. By the Division Algorithm for Polynomials, there exists a unique $q(x)$ and $r(x)$ in $\mathbb{R}[x]$ such that

$$f(x) = g(x)q(x) + r(x)$$

with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x)) = 2$, that is, $r(x)$ is either constant or linear. Substituting $x = c$ into the above gives

$$0 = f(c) = g(c)q(c) + r(c) = r(c)$$

and hence $r(c) = 0$. Assume towards a contradiction that $r(x)$ is linear. By definition, $r(x) = ax + b \in \mathbb{R}[x]$ with $a \neq 0$. Then

$$r(c) = ac + b = 0 \qquad \implies \qquad c = \tfrac{-b}{a} \in \mathbb{R}$$

and this is a contradiction. Therefore, $r(x)$ is a constant polynomial and since $r(c) = 0$, we have that $r(x) = 0$ and thus $g(x) \mid f(x)$. ■

Theorem: (Real Factors of Real Polynomials (RFRP)) Let $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{R}[x]$. Then $f(x)$ can be written as a product of real linear and real quadratic factors,

Proof: By CPN, $f(x)$ has n roots over $\mathbb{C}$. Let $r_1, r_2, ..., r_k$ be the real roots and let $c_1, c_2, ..., c_\ell$ be the strictly complex roots. By CJRT, complex roots come in pairs, say $c_2 = \overline{c_1}$, $c_4 = \overline{c_3}$, ..., $c_\ell = \overline{c_{\ell-1}}$ (hence also ℓ is even). For each pair, by RQF, we have an associated quadratic factor, say $q_1(x), q_2(x), ..., q_{\ell/2}(x)$. By the Factor Theorem, each real root corresponds to a linear factor, say $g_1(x), ..., g_k(x)$. Hence

$$f(x) = cg_1(x)...g_k(x)q_1(x)...q_{\ell/2}(x)$$

where c is the coefficient of the leading term completing the proof. ■

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Prove that a real polynomial of odd degree has a real root.

Solution: Assume towards a contradiction that $p(x)$ is a real polynomial of odd degree without a root. By the Factor Theorem, we know that if $p(x)$ cannot have a real linear factor. By Real Factors of Real Polynomials, we see that

$$p(x) = q_1(x)...q_k(x)$$

for some quadratic factors $q_i(x)$. Now, taking degrees shows that

$$\deg(p(x)) = 2k$$

contradicting the fact that the degree was of $p(x)$ is odd. Hence, the polynomial must have a real root. ■