Lecture 8

Handout or Document Camera or Class Exercise

Consider the following statement.

$$\{2k : k \in \mathbb{N}\} \supseteq \{n \in \mathbb{Z} : 8 \mid (n+4)\}$$

A well written and correct direct proof of this statement could begin with

A) We will show that the statement is true in both directions.

B) Assume that $8 \mid 2n$ where n is an integer.

C) Let $m \in \{n \in \mathbb{Z} : 8 \mid (n+4)\}$.

D) Let $m \in \{2k : k \in \mathbb{N}\}$.

E) Assume that $8 \mid (2k+4)$.

Solution: Let $m \in \{n \in \mathbb{Z} : 8 \mid (n+4)\}$.

Instructor's Comments: This is the 5 minute mark

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Notes:

(i) A single counter example proves that $(\forall x \in S, P(x))$ is false.

Claim: Every positive even integer is composite.

This claim is false since 2 is even but 2 is prime.

(ii) A single example does not prove that $(\forall x \in S, P(x))$ is true.

Claim: Every even integer at least 4 is composite.

This is true but we cannot prove it by saying "6 is an even integer and is composite." We must show this is true for an arbitrary even integer x. (Idea: $2 \mid x$ so there exists a $k \in \mathbb{N}$ such that $2k = x$ and $k \neq 1$.)

(iii) A single example does show that $(\exists x \in S, P(x))$ is true.

Claim: Some even integer is prime.

This claim is true since 2 is even and 2 is prime.

(iv) What about showing that $(\exists x \in S, P(x))$ is false?

Idea: $(\exists x \in S, P(x))$ is false $\equiv \forall x \in S, \neg P(x)$ is true. This idea is central for proof by contradiction which we will see later.

Instructor's Comments: This is the 10-13 minute mark

Negating Quantifiers Example: Negate the following:

(i) Everybody in this room was born before 2010.

Solution: Somebody in this room was not born before 2010.

(ii) Someone in this room was born before 1990

Solution: Everyone in this room was born after 1990.

(iii) $\forall x \in \mathbb{R}, |x| < 5$

Solution: $\neg(\forall x \in \mathbb{R}, |x| < 5) \equiv \exists x \in \mathbb{R}, |x| \geq 5$

(iv) $\exists x \in \mathbb{R}, |x| \leq 5$

Solution: $\neg(\exists x \in \mathbb{R}, |x| \leq 5) \equiv \forall x \in \mathbb{R}, |x| > 5$

Instructor's Comments: Let them validate the truth of the above statements. This could take you to the 20 minute mark easily

Note: A proof that a statement is false is called a disproof.

Example: Prove or disprove: Let $a, b, c \in \mathbb{Z}$. If $a \mid bc$ then $a \mid b$ or $a \mid c$.

Solution: This is false! A counter example is given by $a = 6$, $b = 2$ and $c = 3$. Then $a \mid bc$ BUT $6 \nmid 2$ and $6 \nmid 3$.

Note: It turns out that this is true if you require additionally that a is prime. This is called Euclid's Lemma. We'll see a proof of this in 5 weeks. It is actually very nontrivial to prove.

Instructor's Comments: Get them to think about the prime condition. The proof of this requires GCDs in the prime case to the best of my knowledge. This is the 27 minute mark.

Handout or Document Camera or Class Exercise

Which of the following are true?

(i) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$

(ii) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$

(iv) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$

Solution:

(i) False (Choose $x = y = 0$)

(ii) True (Choose $x = 1$ and $y = 0$)

(iii) True.

Proof: Let $x \in \mathbb{R}$ be arbitrary. then choose $y = \sqrt[3]{x^3 - 1}$. Then

$$x^3 - y^3 = x^3 - (\sqrt[3]{x^3 - 1})^3 = x^3 - (x^3 - 1) = 1$$

(iv) False. Idea: Negate and show the negation is true!

$$\neg(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1) \equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 \neq 1$$

Proof: Let $x \in \mathbb{R}$ be arbitrary. Take $y = x$. Then $x^3 - y^3 = x^3 - x^3 = 0 \neq 1$.

Instructor's Comments: This is the 40 minute mark

Handout or Document Camera or Class Exercise

List all elements of the set:

$$\{n \in \mathbb{Z} : n > 1 \wedge ((m \in \mathbb{Z} \wedge m > 0 \wedge m \mid n) \Rightarrow (m = 1 \vee m = n))\} \cap \{n \in \mathbb{Z} : n \mid 42\}$$

Solution: The first set is the set of all primes. The second set is the set of all divisors of 42, namely

$$\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42\}.$$

The intersection is therefore $\{2, 3, 7\}$.

Check out `http://www.cemc.uwaterloo.ca/~cbruni/Math135Resources.php` for symbol cheat sheets and theorem cheat sheets and other goodies!

Instructor's Comments: End of class