MATHEMATICS 103 Section 206
Series tests

I would like to thank Vince Chan for both the file and the idea of doing this.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ (including if it does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
Note: This works only in one direction. Try this first because it takes little time to apply.

Integral Test

If f is a continuous, positive, and (eventually) decreasing function on $[1, \infty)$ with $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges. That is,

(i) $\int_1^{\infty} f(x)\, dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) $\int_1^{\infty} f(x)\, dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Note: You need to check f is continuous, positive, and (eventually) decreasing. Try this when a_n looks like an integral you can evaluate. This test often works when dealing with logarithms.

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p-Series Test

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$
Note: This works best in conjunction with the Comparison Test below.

Comparison Test

Suppose $0 \leq a_n \leq b_n$. Then

(i) $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges.

Note: Each of these only work in one direction, and both terms must be positive. Try comparing with p-series or geometric series.

Absolute Convergence Test

Absolute convergence is stronger than convergence: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Ratio Test

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. We have:

(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Note: If the limit of the ratio is 1, then the test is inconclusive. This happens whenever you have terms that look like polynomials or logarithms, for instance. This test is ideal for terms that involve products, like exponentials and factorials.