Trigonometric Integration

Goal: Integrate functions involving sines and cosines or tangent and secants.

We take a scaffolding approach to this problem:

(i) $\int (1 - \sin^2(x)) \sin^3(x) \cos(x)\, dx$

(ii) $\int \cos^2(x) \sin^3(x) \cos(x)\, dx$

(iii) $\int \cos^3(x) \sin^3(x)\, dx$

The idea: when a power of either cosine or sine is odd, separate a power of the odd function, then use the substitution $\sin^2(x) + \cos^2(x) = 1$.

What about for even powers? Use double angle formulas!

(i) $\int \frac{1+\cos(2x)}{2}\,dx$

(ii) $\int \cos^2(x)\,dx$

(iii) $\int \frac{1-\cos(2x)}{2}\,dx$

(iv) $\int \sin^2(x)\,dx$

(v) $\int \cos^4(x)\,dx$

Rules for integrating powers of sine and cosine

(i) If the power of $\sin(x)$ (or $\cos(x)$) is odd, split off a power of $\sin(x)$ (or $\cos(x)$) and change the other powers using $\sin^2 = 1 - \cos^2(x)$ (or $\cos^2 = 1 - \sin^2(x)$). Then substitute $u = \cos(x)$ (or $u = \sin(x)$).

For example, if m is an odd integer, then

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{m-1}(x)\cos^n(x)\sin(x)\,dx = \int (1-\cos^2(x))^{(m-1)/2}\cos^n(x)\sin(x)\,dx$$

then let $u = \cos(x)$.

(ii) If both powers are even, then use the double angle formulas.

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

What's the key idea in all this? The key is that $\sin(x)$ and $\cos(x)$ are derivatives of each other! Let's try this with $\tan(x)$ and $\sec(x)$.

(i) $\int \tan^6(x)\sec^2(x)\,dx$

(ii) $\int \tan^6(x)(\tan^2(x)+1)\sec^2(x)\,dx$

(iii) $\int \tan^6(x)\sec^4(x)\,dx$

What about these examples?

(i) $\int (\sec^2(x)-1)^2\sec^2(x)(\tan(x)\sec(x))\,dx$

(ii) $\int \tan^4(x)\sec^2(x)\tan(x)\sec(x)\,dx$

(iii) $\int \tan^5(x)\sec^3(x)\,dx$

Rules for integrating powers of tangent and secant

(i) If the power of $\sec(x)$ is even and not zero, split off a power of $\sec^2(x)$ and change the other powers using $\sec^2(x) = 1 + \tan^2(x)$. Then substitute $u = \tan(x)$.

For example, if m and n are even integers, then

$$\begin{aligned}\int \sec^m(x)\tan^n(x)\,dx &= \int \sec^{m-2}(x)\tan^n(x)\sec^2(x)\,dx \\ &= \int (1+\tan^2(x))^{(m-2)/2}\tan^n(x)\sec^2(x)\,dx\end{aligned}$$

then let $u = \tan(x)$.

(ii) If the power of $\tan(x)$ is odd (and we have at least one copy of $\sec(x)$), then split off a $\tan(x)\sec(x)$ term and substitute the remaining powers of $\tan(x)$ using $\tan^2 = \sec^2(x) - 1$. Then substitute $u = \sec(x)$.

For example, if n is an odd integer and m is at least one, then

$$\begin{aligned}\int \sec^m(x)\tan^n(x)\,dx &= \int \sec^{m-1}(x)\tan^{n-1}(x)\sec(x)\tan(x)\,dx \\ &= \int \sec^{m-1}(x)(\sec^2(x)-1)^{(n-1)/2}\sec(x)\tan(x)\,dx\end{aligned}$$

and finally substitute $u = \sec(x)$.

NB If the powers of $\tan(x)$ and $\sec(x)$ don't fall into the above rules, then the proper strategy varies wildly (recall the integrals of $\tan(x)$ and $\sec(x)$ for example).

NB This idea can be applied to $\csc(x)$ and $\cot(x)$ but I will not elaborate in class.

List of trigonometric identities

(i) $\sin^2(x) + \cos^2(x) = 1$

(ii) $\tan^2(x) + 1 = \sec^2(x)$ (divide the first equation by $\cos^2(x)$).

(iii) $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$. This combines the two identities given by

$$\sin(A+B) = \sin(A)\cos(B)+\sin(B)\cos(A) \quad \text{and} \quad \sin(A-B) = \sin(A)\cos(B)-\sin(B)\cos(A)$$

(iv) $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$. This combines the two identities given by

$$\cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B) \quad \text{and} \quad \cos(A-B) = \cos(A)\cos(B)+\sin(A)\sin(B)$$

(v) $\sin(2x) = 2\sin(x)\cos(x)$

(vi) $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

(vii) $\cos^2(x) = \frac{1+\cos(2x)}{2}$

(viii) $\sin^2(x) = \frac{1-\cos(2x)}{2}$

NB The above are for a general x so for example,

$$\cos(4x) = \cos(2(2x)) = \cos^2(2x) - \sin^2(2x) = 2\cos^2(2x) - 1 = 1 - 2\sin^2(2x)$$

and similarly,

$$\cos^2(2x) = \frac{1+\cos(4x)}{2} \qquad \sin^2(2x) = \frac{1-\cos(4x)}{2}$$