Solving a Series

Carmen Bruni

A Sample Series Problem

Question: Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

First Attempt

First let's think about what this series is - maybe the terms are big and the series will diverge by the divergence test. There's a bunch of positive terms and the terms are decreasing. In fact, if I take the limit the limit as n tends to infinity, I get that

$$\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^5+14n^3+4n} = \lim_{n\to\infty} \frac{n^3(1+3/n+3/n^2)}{n^5(1+14/n^2+4/n^4)}$$
$$= \lim_{n\to\infty} \frac{1+3/n+3/n^2}{n^2(1+14/n^2+4/n^4)} = 0$$

First Attempt

First let's think about what this series is - maybe the terms are big and the series will diverge by the divergence test. There's a bunch of positive terms and the terms are decreasing. In fact, if I take the limit the limit as n tends to infinity, I get that

$$\lim_{n\to\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n} = \lim_{n\to\infty} \frac{n^3(1 + 3/n + 3/n^2)}{n^5(1 + 14/n^2 + 4/n^4)}$$
$$= \lim_{n\to\infty} \frac{1 + 3/n + 3/n^2}{n^2(1 + 14/n^2 + 4/n^4)} = 0$$

... so the divergence test tells us nothing. Remember that series with small terms can diverge; for example $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In fact, when dealing with polynomials when the denominator has higher degree than the numerator, this will never work!

Attempt Two

Well we said that the terms were positive, decreasing, it tends to 0 as n tends to infinity and our function is continuous! Thus we can try to apply the integral test and evaluate the integral! The convergence or divergence of this series is the same as the divergence or convergence as

$$\int_1^\infty \frac{x^3 + 3x^2 + 1}{x^5 + 14x^3 + 4x}\, dx.$$

Attempt Two

Well we said that the terms were positive, decreasing, it tends to 0 as n tends to infinity and our function is continuous! Thus we can try to apply the integral test and evaluate the integral! The convergence or divergence of this series is the same as the divergence or convergence as

$$\int_1^\infty \frac{x^3 + 3x^2 + 1}{x^5 + 14x^3 + 4x}\, dx.$$

... well this looks like a disaster - this integral is far from easy! Instead of struggling through this let's try another approach. Sometimes this can work but...

Attempt Three

...well what's left? Let's try the ratio test! We need to evaluate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Let's try:

$$\lim_{n\to\infty} \frac{(n+1)^3 + 3(n+1)^2 + 1}{(n+1)^5 + 14(n+1)^3 + 4(n+1)} \div \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

Attempt Three

...well what's left? Let's try the ratio test! We need to evaluate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Let's try:

$$\lim_{n\to\infty} \frac{(n+1)^3 + 3(n+1)^2 + 1}{(n+1)^5 + 14(n+1)^3 + 4(n+1)} \div \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

This looks like a mess! Well we can try to flip the second fraction and we can start to pull out factors of n from the first fraction. This gives us

$$\lim_{n\to\infty} \frac{n^3(1+1/n)^3 + 3n^2(1+1/n)^2 + 1}{n^5(1+1/n)^5 + 14n^3(1+1/n)^3 + 4n(1+1/n)} \cdot \frac{n^5 + 14n^3 + 4n}{n^3 + 3n^2 + 1}$$

Attempt Three Continued

Alright now we can factor the largest power in the numerator and the denominator.

$$\lim_{n\to\infty} \frac{n^3((1+1/n)^3 + 3/n(1+1/n)^2 + 1/n^3)}{n^5((1+1/n)^5 + 14/n^2(1+1/n)^3 + 4/n^4(1+1/n))} \cdot \frac{n^5(1+14/n^2+4/n^4)}{n^3(1+3/n+1/n^3)}$$

Attempt Three Continued

Alright now we can factor the largest power in the numerator and the denominator.

$$\lim_{n\to\infty} \frac{n^3((1+1/n)^3 + 3/n(1+1/n)^2 + 1/n^3)}{n^5((1+1/n)^5 + 14/n^2(1+1/n)^3 + 4/n^4(1+1/n))} \cdot \frac{n^5(1+14/n^2+4/n^4)}{n^3(1+3/n+1/n^3)}$$

Ha! Now we can cancel factors and plug in the limit!

$$\lim_{n\to\infty} \frac{(1+1/n)^3 + 3/n(1+1/n)^2 + 1/n^3}{(1+1/n)^5 + 14/n^2(1+1/n)^3 + 4/n^4(1+1/n)} \cdot \frac{1+14/n^2+4/n^4}{1+3/n+1/n^3} = 1$$

Attempt Three Continued

Alright now we can factor the largest power in the numerator and the denominator.

$$\lim_{n\to\infty} \frac{n^3((1+1/n)^3 + 3/n(1+1/n)^2 + 1/n^3)}{n^5((1+1/n)^5 + 14/n^2(1+1/n)^3 + 4/n^4(1+1/n))} \cdot \frac{n^5(1+14/n^2+4/n^4)}{n^3(1+3/n+1/n^3)}$$

Ha! Now we can cancel factors and plug in the limit!

$$\lim_{n\to\infty} \frac{(1+1/n)^3 + 3/n(1+1/n)^2 + 1/n^3}{(1+1/n)^5 + 14/n^2(1+1/n)^3 + 4/n^4(1+1/n)} \cdot \frac{1+14/n^2+4/n^4}{1+3/n+1/n^3} = 1$$

Gosh darn it the ratio test is useless here! So not only did we spend a lot of time trying it but it did not help. In fact the ratio test will never help for ratios of polynomials!

Attempt Four

- Alright we're running out of options. The terms are all positive so let's try the comparison test (we could have tried the integral comparison test while attempting to apply the integral test but I decided to save it for now). First we need some sort of feeling for if this sum converges or diverges.

Attempt Four

- Alright we're running out of options. The terms are all positive so let's try the comparison test (we could have tried the integral comparison test while attempting to apply the integral test but I decided to save it for now). First we need some sort of feeling for if this sum converges or diverges.
- Let's look at the highest power in the numerator and denominator. The numerator has a n^3 term and the denominator has a n^5 term. So the ratio of these terms is $n^3/n^5 = 1/n^2$. If we think about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, well - we get convergence! Why? This is just an application of the p-series test! Therefore, intuitively, this series should converge by the p series test and comparison test. Now we just have to make it happen.

Attempt Four Continued

Time to look at the question again.

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

To make this happen, we need to bound the numerator above by some factor of n^3 and bound the denominator below by some factor of n^5.

Attempt Four Continued

Time to look at the question again.

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

To make this happen, we need to bound the numerator above by some factor of n^3 and bound the denominator below by some factor of n^5. Bounding the numerator is easy! Each term is less than n^3 so we just replace the terms with n^3 terms.

$$n^3 + 3n^2 + 1 < n^3 + 3n^3 + n^3 = 5n^3$$

Attempt Four Continued

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n}$$

Bounding the polynomial in the denominator from below is also easy! The other terms of positive and so forgetting about these terms makes the polynomial smaller!

$$n^5 < n^5 + 14n^3 + 4n$$

This lower bound implies that

$$\frac{1}{n^5 + 14n^3 + 4n} < \frac{1}{n^5}$$

Attempt Four Final Slide

We can bound our sum via

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n^2 + 1}{n^5 + 14n^3 + 4n} < \sum_{n=1}^{\infty} \frac{5n^3}{n^5} < 5 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the last sum converges by the p series test with $p = 2$! Hence our original sum converges by the comparison test. (As a final note, the limit comparison test would also work if you know this technique). For polynomials, the comparison test is often your best bet.