Series Part 2

Carmen Bruni

Review

So far we have a few tests and methods to evaluate series

- To evaluate series: Geometric Series and Telescoping Series techniques.
- Divergence test (adding big numbers gives a big answer)
- Integral test (series are integrals)
- p-series test (monomials are simple)

Today we complete our list by adding two more techniques, namely the comparison test and the ratio test.

Recall that folded piece of paper from class. The second part to the paper contained the integral comparison test for improper integrals which stated that

Theorem (Integral Comparison Test)

Suppose $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ with $0 \le g(x) \le f(x)$ on this domain. Then

i) *If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.*
ii) *If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.*

The example we did in class showed $\int_a^\infty e^{-x^2}\,dx < \int_a^\infty e^{-x}\,dx$ and the latter converged so the former had to converge as well by the comparison test.

Back to series

Perhaps unsurprisingly given our discussion on the integral test, this test will carry over to series.

Theorem

Suppose $0 \leq a_n \leq b_n$ *(eventually). Then*

(i) *If* $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges.*

Example

Recall from last class the example $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Example

Recall from last class the example $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Notice that for $n \geq 3$ we have

$$\frac{1}{n} \leq \frac{\ln(n)}{n}.$$

Example

Recall from last class the example $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Notice that for $n \geq 3$ we have

$$\frac{1}{n} \leq \frac{\ln(n)}{n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series. Thus, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges by the comparison test.

Example

Let's do a "polynomial" example. Say $\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1}$. Notice that this is a polynomial with all positive terms. So let's try to use the comparison test.

Example

Let's do a "polynomial" example. Say $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1}$. Notice that this is a polynomial with all positive terms. So let's try to use the comparison test.

Intuition: Since the leading term (the term with the biggest power of n) in the top is $\sqrt[3]{2n}$ and the leading term in the denominator is n^5, we should believe that

$$\frac{\sqrt[3]{2n-1}}{n^5+2n+1} \approx \frac{\sqrt[3]{2n}}{n^5} = \frac{\sqrt[3]{2}}{n^{14/3}}$$

and the latter converges by the p series test so our original series should also converge by the comparison test.

Example

Let's do a "polynomial" example. Say $\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1}$. Notice that this is a polynomial with all positive terms. So let's try to use the comparison test.

Intuition: Since the leading term (the term with the biggest power of n) in the top is $\sqrt[3]{2n}$ and the leading term in the denominator is n^5, we should believe that

$$\frac{\sqrt[3]{2n-1}}{n^5+2n+1} \approx \frac{\sqrt[3]{2n}}{n^5} = \frac{\sqrt[3]{2}}{n^{14/3}}$$

and the latter converges by the p series test so our original series should also converge by the comparison test. Note: While the intuition is good, it alone will not earn full marks on a test. You need to be more formal. The intuition however can *guide* your proof. Since we believe our series converges, we should try to find an upper bound on the fraction. This reduces to finding an upper bound on the numerator and a lower bound on the denominator.

Example

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1}$.

An upper bound on the numerator is given by

$$\sqrt[3]{2n-1} < \sqrt[3]{2n}$$

Example

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1}$.

An upper bound on the numerator is given by

$$\sqrt[3]{2n-1} < \sqrt[3]{2n}$$

A lower bound on the denominator is given by

$$n^5 + 2n + 1 > n^5 \quad \Rightarrow \quad \frac{1}{n^5+2n+1} < \frac{1}{n^5}$$

Comparing these, we have

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n^5+2n+1} < \sum_{n=1}^{\infty} \frac{\sqrt[3]{2n}}{n^5} = \sum_{n=1}^{\infty} \frac{\sqrt[3]{2}}{n^{14/3}}$$

As $14/3 > 1$, the latter converges by the p-series test. Thus the sum converges by the comparison test.

Another example

What about $\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n+\sqrt{n}+3}$. Intuition says that the terms should look like $\frac{\sqrt[3]{2n}}{n} = \frac{\sqrt[3]{2}}{\sqrt[3]{n^2}}$ and so this series should diverge. Thus, we need to bound the fractions from below.

Another example

What about $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n+\sqrt{n}+3}$. Intuition says that the terms should look like $\frac{\sqrt[3]{2n}}{n} = \frac{\sqrt[3]{2}}{\sqrt[3]{n^2}}$ and so this series should diverge. Thus, we need to bound the fractions from below. A lower bound on the numerator is given by

$$\sqrt[3]{2n-1} \geq \sqrt[3]{2n-n} = \sqrt[3]{n}$$

Another example

What about $\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n+\sqrt{n}+3}$. Intuition says that the terms should look like $\frac{\sqrt[3]{2n}}{n} = \frac{\sqrt[3]{2}}{\sqrt[3]{n^2}}$ and so this series should diverge. Thus, we need to bound the fractions from below. A lower bound on the numerator is given by

$$\sqrt[3]{2n-1} \geq \sqrt[3]{2n-n} = \sqrt[3]{n}$$

An upper bound on the denominator is given by

$$n+\sqrt{n}+3 \leq n+n+3n = 5n \quad \Rightarrow \quad \frac{1}{n+\sqrt{n}+3} \geq \frac{1}{5n}$$

Comparing these, we have

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n-1}}{n+\sqrt{n}+3} \geq \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{5n} = \sum_{n=1}^{\infty} \frac{1}{5n^{2/3}}$$

As $2/3 \leq 1$, the latter diverges by the p-series test. Thus the sum diverges by the comparison test.

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? What does our intuition say?

A Our series is like $\frac{1}{n^3}$ so it converges.

B Our series is like $\frac{1}{n^3}$ so it diverges.

C Our series is like $\frac{1}{n^2}$ so it converges.

D Our series is like $\frac{1}{n^2}$ so it diverges.

E Other

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? What does our intuition say?

A Our series is like $\frac{1}{n^3}$ so it converges.

B Our series is like $\frac{1}{n^3}$ so it diverges.

C Our series is like $\frac{1}{n^2}$ so it converges.

D Our series is like $\frac{1}{n^2}$ so it diverges.

E Other

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? To proceed we should expect to use

A The divergence test

B The p-series test

C The integral test

D The comparison test

E Some combination of the above

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? To proceed we should expect to use

A The divergence test

B The p-series test

C The integral test

D The comparison test

E Some combination of the above

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? Which of the following are we required to do to set out our plan?

A Bound the numerator above

B Bound the numerator below

C Bound the denominator above

D Bound the denominator below

E Some combination of the above

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? Which of the following are we required to do to set out our plan?

A Bound the numerator above

B Bound the numerator below

C Bound the denominator above

D Bound the denominator below

E Some combination of the above

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? Which of the following proposed lower bounds achieves our goal?

A $2n^3$

B n^2

C $n^3 + 1$

D $3n^3$

Example for the class to try

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge? Which of the following proposed lower bounds achieves our goal?

A $2n^3$

B n^2

C n^3+1

D $3n^3$

Solution

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge?

Solution

Question: Does the series $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converge?

Solution: Since $2n^3 < 2n^3 + 1$ for all $n \geq 1$, we have that

$$\sum_{n=1}^{\infty} \frac{n}{2n^3+1} < \sum_{n=1}^{\infty} \frac{n}{2n^3} = \sum_{n=1}^{\infty} \frac{1}{2n^2}$$

As the right most series above converges by the p-series test (since $2 > 1$) we have that our given series must converge by the comparison test, a valid use since all our terms are positive.

Notation

Definition

Define the factorial notation by $0! = 1$, $1! = 1$ and $n! = n(n-1)(n-2)...(2)(1)$.

Notation

Definition

Define the factorial notation by $0! = 1$, $1! = 1$ and $n! = n(n-1)(n-2)...(2)(1)$.

So for example $3! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4(6) = 24$.

Notation

Definition

Define the factorial notation by $0! = 1$, $1! = 1$ and $n! = n(n-1)(n-2)...(2)(1)$.

So for example $3! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4(6) = 24$.

Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Definition

Define the factorial notation by $0! = 1$, $1! = 1$ and $n! = n(n-1)(n-2)...(2)(1)$.

So for example $3! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4(6) = 24$.

Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem

An absolutely convergent series is itself convergent.

Ratio Test

The ratio test behaves much like our geometric series. It says if adjacent terms have a ratio that is, in the limit, large, then your series should diverge. Conversely, if the ratios are small then our series should converge. If the terms are close to each other in size, then this test tells us little.

Ratio Test

Theorem

Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. *We have:*

(i) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(ii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$ *then the test is inconclusive.*

Note: If the limit of the ratio is 1, then the test is inconclusive. This happens whenever you have terms that look like polynomials or logarithms, for instance. This test is ideal for terms that involve products, like exponentials and factorials.

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n} \right|$$

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n}\right| \end{aligned}$$

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n}\right| \\
&= \lim_{n\to\infty} \left|\frac{1+1/n}{7}\right|
\end{aligned}$$

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{1+1/n}{7}\right| \\
&= \frac{1}{7}
\end{aligned}$$

Example

Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n} \right| \\
&= \lim_{n\to\infty} \left| \frac{1+1/n}{7} \right| \\
&= \frac{1}{7} \\
&< 1
\end{aligned}$$

Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converge? Solution: Use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{7^{n+1}} \cdot \frac{7^n}{n} \right| \\
&= \lim_{n\to\infty} \left| \frac{1+1/n}{7} \right| \\
&= \frac{1}{7} \\
&< 1
\end{aligned}$$

Thus our series converges by the ratio test.

Example for the class to try

Does $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge?

A Yes, by the ratio test since the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is 1.

B Yes, by the ratio test since the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is 0.

C Yes, by the ratio test since the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is ∞.

D No, by the ratio test since the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is ∞

E No, by the ratio test since the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is 1

Solution

Does $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Solution

Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right|$$

Solution

Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\ &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \end{aligned}$$

Solution

Does $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)!} \right|
\end{aligned}$$

Solution

Does $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)n!} \right|
\end{aligned}$$

Solution

Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2}{n+1} \right|
\end{aligned}
$$

Solution

Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2}{n+1} \right| \\
&= 0 < 1
\end{aligned}$$

Solution

Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge? Solution: Use the ratio test to see

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2 \cdot n!}{(n+1)n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{2}{n+1} \right| \\
&= 0 < 1
\end{aligned}$$

Thus our series converges by the ratio test.

Series tips

Rules of thumb:

- Always try the divergence test first because it is usually easy to try (after a while you'll just do it in your head and only write it down when it works). Eg. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$
- For series with logarithms, usually the integral test works. Eg. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$
- For series with (only) polynomials, try the comparison test coupled with the p series test. Eg. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+2n+10}$.
- For series with exponentials or factorials, try the ratio test.
- For series involving negative terms, try to test for absolute convergence.

Determine which of the following series converges

1. $$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$
2. $$\sum_{n=1}^{\infty} \arctan(n)$$
3. $$\sum_{n=1}^{\infty} \frac{n!}{3^n}$$
4. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$
5. $$\sum_{n=2}^{\infty} \frac{n^2 - 1}{\sqrt{3n^4 + 1}}$$

Next week

Next class: The march towards Taylor series. REMINDER! Teaching Evaluations! Please vote

UBC site

RMP