Series

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Review

Recall from class that an infinite series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{S_N\}$ converges where

$$S_N = \sum_{n=1}^{N} a_n.$$

In the case of convergence, we write

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N.$$

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Recall that series are linear, that is

$$\sum_{n=1}^{\infty}(a_n + cb_n) = \sum_{n=1}^{\infty} a_n + c\sum_{n=1}^{\infty} b_n$$

provided both series on the right converge.

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We already discussed two important classes of convergent series. The first of which were geometric series. These were series of the form

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When $|r| < 1$, we saw that the infinite series converged and gave

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Geometric Series

Recall: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

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B 9/25

C 3/22

D 3/25

E Other

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Our a value is $\frac{3^2}{5^2} = \frac{9}{25}$. The common ratio is given by

$$r = \frac{\frac{3^3}{5^4}}{\frac{3^2}{5^2}} = \frac{3}{5^2} = \frac{3}{25}.$$

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Thus,

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^{2n}} = \frac{9/25}{1 - 3/25} = \frac{9/25}{22/25} = \frac{9}{22}.$$

Telescoping series

The other type of series discussed were telescoping series. These series gave us amazing cancellation properties to get our desired result. For example

$$\sum_{n=1}^{\infty}(e^{-n} - e^{-n+1})$$

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&\quad = \lim_{N\to\infty} \sum_{n=1}^{N}(e^{-n} - e^{-n+1}) \\
&\quad = \lim_{N\to\infty}((e^{-1} - e^{0}) + (e^{-2} - e^{-1}) + \ldots(e^{-N} - e^{-N+1})) \\
&\quad = \lim_{N\to\infty}((-e^{0} + e^{-1}) + (-e^{-1} + e^{-2}) + \ldots(-e^{-N+1} + e^{-N})) \\
&\quad = \lim_{N\to\infty}((-e^{0} + \cancel{e^{-1}) + (-e^{-1} + e^{-2}) + \ldots(-e^{-N+1}} + e^{-N})) \\
&\quad = -1
\end{aligned}$$

Example: Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. (Hint: Use partial fractions)

A 1

B $1/n$

C $3/2$

D $1/2$

E Other

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$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} ((1 - 1/2) + (1/2 - 1/3) + \ldots(1/N - 1/(N+1)) \\
&= \lim_{N\to\infty} ((1 \cancel{- 1/2) + (1/2 - 1/3) + \ldots(1/N} - 1/(N+1)) \\
&= 1
\end{aligned}$$

When does a series converge

Sometimes it is clear from the partial sums that a series diverges. For example

$$\sum_{n=1}^{\infty} n$$

Examining the partial sums gives

$$S_1 = 1 \qquad S_2 = 1 + 2 = 3 \qquad S_3 = 1 + 2 + 3 = 6$$

and in general

$$S_N = 1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

This limit tends to infinity.

When does a series converge

- We already saw that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1.$
- I mentioned earlier that miraculously, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$
- What about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$? Does this series converge or diverge?

A Converges

B Diverges

Divergence Test

In practice, coming up with exact values for sums is hard. However we can often deduce when a series converges or diverges by using some nice rules of thumb. The first of which is the divergence test. Intuitively if the terms are not small to begin with, then the series cannot converge

Theorem (Divergence Test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Reworded, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Does this help?

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What does this say about the convergence of $\sum_{n=1}^{\infty} \frac{1}{n}$?

A Converges by the divergence test.

B Diverges by the divergence test.

C Cannot determine using the divergence test.

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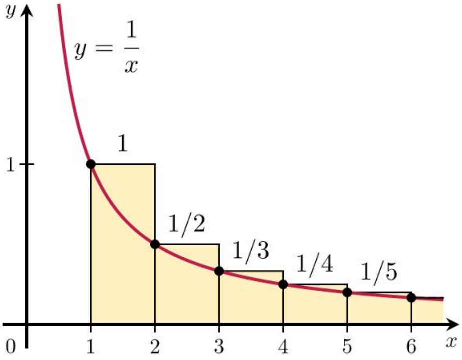
Solution: Since the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0, the divergence test tells us nothing about the convergence of the series.

Integrals to the rescue!

Let's examine this picture.



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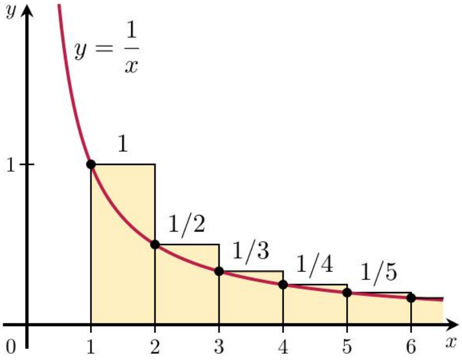

Using the left Riemann sum, we obtain an overestimate of the integral using our series $\sum_{n=1}^{\infty} \frac{1}{n}$. Using the right endpoints, we would obtain a lower estimate. This gives another test.

Theorem (Integral Test)

If f is a continuous, positive, and (eventually) decreasing function on $[1,\infty)$ with $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges. That is,

(i) $\int_1^{\infty} f(x)\,dx$ *converges* $\Rightarrow \sum_{n=1}^{\infty} a_n$ *converges.*

(ii) $\int_1^{\infty} f(x)\,dx$ *diverges* $\Rightarrow \sum_{n=1}^{\infty} a_n$ *diverges.*

Note: The value of the series is NOT [in general] equal to the value of the associated integral. Test this out for $\int_1^{\infty} \frac{dx}{x^2}$.

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Solution: Let $f(x) = 1/x$, a positive, continuous and decreasing function on $[1, \infty)$. Since $\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \ln|b|$, we see that the integral and hence the series diverges by the integral test.

Fun Fact

In fact, this tells us how quickly our series grows as well! Define the Euler-Mascheroni constant as

$$\gamma = \lim_{N\to\infty}\left(\left(\sum_{n=1}^{N}\frac{1}{n}\right) - \ln(N)\right) = 0.5772156649...$$

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Open Question: Is γ rational?

More tests!

Recall the folded piece of paper from class: When does $\int_1^\infty x^q \, dx$ converge? Above we showed it diverges when $q = -1$ and we have when $q \neq -1$ that

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$$\begin{aligned}\int_1^\infty x^q \, dx &= \lim_{b\to\infty} \int_1^b x^q \, dx \\ &= \lim_{b\to\infty} \left(\frac{x^{q+1}}{q+1} \right) \Big|_1^b \\ &= \lim_{b\to\infty} \left(\frac{b^{q+1}}{q+1} - \frac{1}{q+1} \right) \\ &= \begin{cases} \frac{-1}{q+1} & \text{if } q+1 < 0 \\ \text{diverges} & \text{if } q+1 > 0. \end{cases}\end{aligned}$$

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Rewording, we have $\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \leq 1$ (notice these are different than the above since here I moved x^p to the denominator).

This gives another test!

Theorem (p-series test)

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ *and diverges if* $p \leq 1$

Two examples

Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ and let $T = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{9973}}$. Then from the p-series test, we can conclude that

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Determine which of the following series converges

1. $$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$
2. $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$
3. $$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 3}$$
4. $$\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$$
5. $$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

One Final Joke

A mathematician organizes a lottery in which the prize is an infinite amount of money. When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: “1 dollar now, $1/2$ dollar next week, $1/3$ dollar the week after that...”

Next class: The comparison test and the ratio test. Also teaching evaluations.