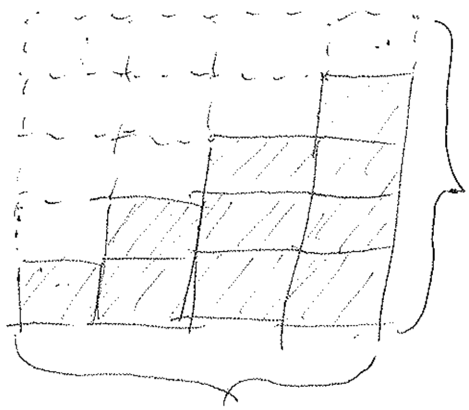
Last time:
Sigma Notation

This time
- Properties of Sigma notation.
- Geometric sums.
- Infinite Series.

Show $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Illustrate for $n=4$.



(Gauss).

$$\text{Area} = n(n+1) = 2\sum_{i=1}^{n} i \Rightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

$$\begin{array}{l} 1+2+3+4+\cdots+n \\ \underline{n+(n-1)+(n-2)+\cdots+1} \\ (n+1)+(n+1)+\cdots\quad (n+1) \leftarrow n \text{ copies.} \end{array}$$

$$\sum_{i=1}^{n} i + \sum_{i=1}^{n}(n-i+1) = n(n+1)$$

same. $2\sum_{i=1}^{n} i = n(n+1)$

Find $\sum_{i=1}^{n} k^2$. Trick: add $\sum_{k=1}^{n}\left((k+1)^3-k^3\right)$

$$\sum_{k=1}^{n}\left((k+1)^3-k^3\right) = \cancel{2^3}-1^3+\cancel{3^3}-\cancel{2^3}+\cancel{4^3}-\cancel{3^3}+5^3-\cancel{4^3}+\cdots+\cancel{n^3}+\cdots(n-1)^3+(n+1)^3-\cancel{n^3}$$

$$= (n+1)^3-1$$

$$= n^3+3n^2+3n.$$

Telescoping sum.

$$\sum_{k=1}^{n}\left((k+1)^3-k^3\right) = \sum_{k=1}^{n}(3k^2+3k+1)$$

$$= 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 \quad \text{(Linearity).}$$

$$\underset{k=1}{1} + \underset{k=2}{1} + \cdots + \underset{k=n}{1}$$

$$n^3+3n^2+3n \;\; 2n = 3\sum_{k=1}^{n}k^2 + 3\frac{n(n+1)}{2} + n$$

$$n(n^2+3n+2) - 3\frac{n(n+1)}{2} = 3\sum_{k=1}^{n}k^2 \quad \text{(isolate \& factor)}$$

$$n(n+2)(n+1) - 3\frac{n(n+1)}{2} = \text{''} \quad \text{(factor)}$$

$$n(n+1)\left(n+2-\frac{3}{2}\right) = \text{''} \quad \text{(common factor).}$$

$$\frac{n(n+1)(2n+1)}{2} = 3\sum_{k=1}^{n}k^2 \quad \left(n+2-\frac{3}{2} = n+\frac{1}{2} = \frac{2n+1}{2}\right)$$

$$\frac{n(n+1)(2n+1)}{6} = \sum_{k=1}^{n}k^2 \quad (*)$$

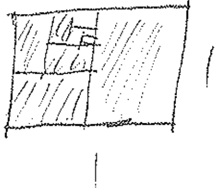
Answer to 100-disc Tower of Hanoi

By formula $(*)$

$$V = \sum_{i=1}^{100}\pi i^2 \cdot h = \sum_{i=1}^{100}\pi i^2(1) = \pi\sum_{i=1}^{100}i^2 = \frac{\pi\, 100(101)(201)}{6}$$

$$= 338350\,\pi.$$

Geometric Series



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = 1$$

Let $r \neq 1$ be a real number (called the common ratio). Let $a \neq 0$ be a real number (the first term).

Then a <u>geometric series</u> is of the form

$$S_N = a + ar + ar^2 + \cdots + ar^N = \sum_{i=0}^{N} ar^i$$

<u>NB</u>: $r = \frac{ar}{a} = \frac{ar^2}{ar} = \cdots = \frac{ar^i}{ar^{i-1}}$ for any i from 1 to N.
Nota bene.

Ex: $S = 1+3+9+27+81+243+729 = \sum_{i=0}^{6} (1)\cdot 3^i = \frac{1\cdot(1-3^7)}{1-3}$

This is a geometric series

Ex: $S = 1+2+3+4+5$ is <u>NOT</u> a geometric series since $\frac{2}{1} \neq \frac{3}{2}$

<u>Claim</u>: $S_N := \sum_{i=0}^{N} ar^i = \frac{a(1-r^{N+1})}{1-r}$

Pf: $S_N = a + ar + ar^2 + \cdots + ar^{N-1} + ar^N$

$rS_N = ar + ar^2 + ar^3 + \cdots + ar^{N-1} + ar^N + ar^{N+1}$

subtract.

$(1-r)S_N = a - ar^{N+1} \Longrightarrow S_N = \frac{a - ar^{N+1}}{1-r} = \frac{a(1-r^{N+1})}{1-r}$ $\boxtimes$

valid since $r \neq 1$.

~~Ex~~ Ex:

$$\sum_{k=1}^{9} 3\cdot 2^{k} \overset{j=k-1}{=} \sum_{j=0}^{8} 3\cdot 2^{j+1} = \sum_{j=0}^{8} 3\cdot 2\cdot 2^{j} = \sum_{j=0}^{8} 6\cdot 2^{j}$$

$a=6$

$r=2$

$N=8$

$$\leadsto = \frac{6(1-2^{8+1})}{1-2}$$

$$= 6(2^{9}-1)$$

$$= 6\cdot 511$$

$$= 3066$$

$$\sum_{k=2}^{9} 3\cdot 2^{k} = \sum_{k=0}^{9} 3\cdot 2^{k} - \sum_{k=0}^{1} 3\cdot 2^{k}$$

Infinite Sums.

Q: How do we add infinitely many numbers.

A: Add finitely many numbers in the begining (the head of the sequence). and take the limit. That is

$$\sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \sum_{i=1}^{N} a_i = \lim_{N\to\infty} S_N$$

where here $S_N = \sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_N$. is called a partial sum.

We say that an infinite series <u>converges</u> if $\lim_{N\to\infty} S_N$ exists (and is finite) and is <u>divergent</u> otherwise

Ex: $$\sum_{k=0}^{\infty} 2^k = \lim_{N\to\infty} \boxed{\sum_{k=0}^{N} 2^k} = \lim_{N\to\infty} \frac{(1)(1-2^{N+1})}{1-2} = \frac{1-\boxed{\lim_{N\to\infty} 2^{N+1}}}{1-2}$$ infinite. DNE.

Thus the sum diverges.

Ex $$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \lim_{N\to\infty} \sum_{k=0}^{N} \left(\frac{1}{2}\right)^k = \lim_{N\to\infty} \frac{(1)\left(1-\left(\frac{1}{2}\right)^{N+1}\right)}{1-\frac{1}{2}} = \frac{1-\lim_{N\to\infty}\left(\frac{1}{2}\right)^{N+1}}{1-\frac{1}{2}}$$
$$= \frac{1-0}{1-\frac{1}{2}} = 2.$$

Thus $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$. Thus $1+\frac{1}{2}+\frac{1}{4}+\cdots = 2$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1$ square halving.

In general

$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} \sum_{n=0}^{N} ar^n = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r}$$

$$= \frac{a(1-\lim_{N\to\infty} r^{N+1})}{1-r}$$

<u>Q</u>: When does this sum converge?

Ans: Converges whenever $\lim_{N\to\infty} r^{N+1}$ converges. (ie limit exists & is finite).

If $|r|>1$ this limit diverges
If $|r|=1$ Recall our $r\neq 1$ so this means $r=-1$
$\lim_{N\to\infty} (-1)^{N+1}$ diverges.
If $|r|<1$ then this converges! Further $\lim_{N\to\infty} r^{N+1}=0$ in this case.

Thus, when $|r|<1$

$$\sum_{n=0}^{\infty} ar^n = \frac{a(1-\lim_{N\to\infty} r^{N+1})}{1-r} = \frac{a}{1-r}.$$

Ex: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1)}{1-\frac{1}{3}} = \frac{1}{2/3} = \frac{3}{2}$.

Thought experiment: Does

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$ converge, <u>OR</u> diverge???