Indefinite Integrals

From before, recall that

$$\int_a^b f(x)\,dx = \text{signed area} = \text{number}$$

these were called **definite integrals**. Now, we will also use the similar notation given by

$$\int f(x)\,dx = \text{antiderivative of } f(x) = \text{family of functions}$$

and we will call these types of integrals **indefinite integrals**. For example

(i) $\displaystyle\int x^2\,dx$

(ii) $\displaystyle\int \frac{dx}{1+x^2}$

NB: The linearity property of the definite integral still holds for indefinite integrals. For example

$\displaystyle\int (2x+1)\,dx$

Integration by Substitution for Indefinite Integrals

Goal: Compute $\displaystyle\int (x+1)^{10}\,dx$

As of right now, we have a very limited toolbox to solve this problem. We either have to guess the integral or expand it out and integrate term by term. Notice here that if $f(x) = x^{10}$ and $g(x) = x+1$, then $f(g(x)) = (x+1)^{10}$. In some sense, we can view this as a problem of undoing the **chain rule** from differential calculus.

Abstraction

Let $F(x)$ and $g(x)$ be differentiable functions. Let $u = g(x)$. The chain rule states that

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

Antidifferentiating (integrating) both sides gives

$$\begin{aligned}\int F'(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx = F(g(x)) + C = F(u) + C = \int \frac{d}{du}F(u)\,du \\ &= \int F'(u)\,du.\end{aligned}$$

This gives

Theorem (Integration by Substitution for Indefinite Integrals)**.** *If $u = g(x)$ is differentiable with continuous derivative, range I an interval and $f(x)$ a continuous function on I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

NB Once you have substituted and solved the integral, you must back substitute (that is, your answer should be in terms of the original variable).

Let's see some examples.

(i) $\displaystyle\int (x+1)^{10}\,dx$

(ii) $\displaystyle\int \frac{x}{1+x^2}\,dx$

(iii) $\displaystyle\int \cot(x)\,dx$

What about definite integrals?

Integration by Substitution for Definite Integrals

Theorem (Integration by Substitution for Definite Integrals)**.** *If $u = g(x)$ is differentiable with continuous derivative, range I an interval and $f(x)$ a continuous function on I, then for $a, b \in \mathbb{R}$, we have*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Notice that here our **endpoints change**. Let's see an example.

$\displaystyle\int_0^\pi \sin(x)e^{\cos(x)}\,dx$

WARNING

(i) With the substitution rule for indefinite integrals, after plugging in your value for u, you **MUST** plug in the original variable back in.

(ii) With the substitution rule for definite integrals, after plugging in your value for u, you **MUST** also change your **endpoints** to match the variable u. You **DO NOT** plug back in your original variable.

Here are some examples of breaking this rule as well as the correct way to write this problem.

Incorrect Examples

For the following, let $u = 1 + x^3$ so that $du = 3x^2dx$ and that

$$u(0) = 1 + (0)^3 = 1 \qquad u(1) = 1 + (1)^3 = 2$$

.

Bad example 1:

$$\int_0^1 \frac{3x^2dx}{1+x^3} = \underbrace{\int_0^1 \frac{du}{u}}_{\text{Bad - Must change endpoints!}} = \ln|u|\Big|_0^1 = \underbrace{\ln|1+x^3|\Big|_0^1}_{\text{Bad - Should have changed endpoints and NOT back substituted}}$$

$$= \ln(2) - \ln(1) = \ln(2)$$

Bad example 2:

$$\int_0^1 \frac{3x^2dx}{1+x^3} = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \underbrace{\ln|1+x^3|\Big|_1^2}_{\text{Bad - do NOT back substitute}}$$

$$= \ln(9) - \ln(2)$$

Good example 1:

$$\int_0^1 \frac{3x^2}{1+x^3}\,dx = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$$

Good example 2:

$$\int \frac{3x^2}{1+x^3}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|1+x^3| + C$$

Using the substitution rule seems easy however there are many cases when applying a substitution is quite unclear. Here are some examples.

Trick 1: Multiplying by 1 (Quite tricky - you will often need a hint to proceed with this type of problem)

$$\int \sec(x)\, dx$$

Trick 2: Perfect square in denominator

$$\int \frac{dx}{x^2 - 6x + 9}$$

Trick 3: Completing the square

$$\int \frac{dx}{x^2 - 6x + 18}$$

More Examples

(i) $\displaystyle\int 2xe^{x^2}\,dx$

(ii) $\displaystyle\int x^5\sqrt{1+x^2}\,dx$

(iii) $\displaystyle\int_5^6 x\sqrt{x-5}\,dx$

(iv) $\displaystyle\int \frac{dx}{x \ln x}$

(v) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}$

(vi) $\displaystyle\int \frac{\ln x}{x}\, dx$