Integration by Parts

Goal: Compute $\int xe^x \, dx$

Notice here that substitution will not help us get to the answer. We need a new technique. In this case, the rule we would really like to undo is the **product rule**. Let's attempt to do this. First

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$$
$$d(uv) = udv + vdu.$$

Antidifferentiating (integrating) both sides gives

$$f(x)g(x) = \int f(x)g'(x) \, dx + \int g(x)f'(x) \, dx$$
$$uv = \int udv + \int vdu$$

and a quick rearrangement gives us the following theorem.

Theorem (Integration by Parts)**.** *Let $f(x) = u$ and $g(x) = v$ be differentiable functions. Then*

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int g(x)f'(x) \, dx$$
$$\int udv = uv - \int vdu$$

The only tricky part is deciding which function to set as our u and as our dv. For example,

$\int xe^x \, dx$

If however we choose the other combination, we get

$\int xe^x \, dx$

which leaves us in a worse position than we started.

Next, here is an example with endpoints. For integration by parts, we simply leave the endpoints the same throughout.

$$\int_0^\pi x\cos(x)\,dx$$

There is a rule of thumb we can follow to help us choose which function the u should be and which the dv should be.

(i) **L** Logarithms. For example $\ln x$

(ii) **I** Inverse trig functions. For example $\arctan x, \arcsin x$

(iii) **A** Algebraic functions. For example $x^3, 3x$

(iv) **T** Trig functions. For example $\sin x \cos x$

(v) **E** Exponentials. For example e^x

Let's look at some more examples. Notice that even similar looking problems can require vastly different techniques!

(i) **Trick 1:** Integrating by parts setting $dv = dx$.

$$\int \arctan(x)\,dx$$

(ii) **Trick 2:** Integrating by parts twice.

$$\int t^2 e^t \, dt$$

(iii) **Trick 3:** Integrals that cycle

$$\int e^x \sin(x) \, dx$$

Let's try some examples with logarithms.

(i) $\displaystyle\int \ln(x)\,dx$

(ii) $\displaystyle\int x^6 \ln(x)\,dx$

(iii) $\displaystyle\int \frac{\ln(x)}{x}\,dx$

(iv) $\displaystyle\int x^{-2} \ln(x)\,dx$