Integration by Partial Fractions

Recall:

$$\int \frac{dx}{x+a} = \ln|x+a| + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{(\frac{x}{a})^2+1} = \frac{1}{a^2}(a\arctan(x/a)) + C = \frac{\arctan(x/a)}{a} + C$$

Key idea:

$\displaystyle\int \frac{3x+1}{x^2+2x-3}\,dx$ This seems like a hard integral. However...

$\displaystyle\int \left(\frac{1}{x-1} + \frac{2}{x+3}\right) dx = \ln|x-1| + 2\ln|x+3| + C$ is an easy integral!

Notice that

$$\frac{1}{x-1} + \frac{2}{x+3} = \frac{x+3+2(x-1)}{(x-1)(x+3)} = \frac{x+3+2x-2}{x^2+2x-3} = \frac{3x+1}{x^2+2x-3}$$

so these integrals are the same thing! The question now becomes how do we go from an 'ugly' rational polynomial function

$$f(x) = \frac{P(x)}{Q(x)}$$

to a 'nice' sum of rational function? We follow these four steps

(i) If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$ then do long division (in MATH 103 we will not require this step)

(ii) Simplify into smaller pieces depending on the factorization of the denominator. For example, if

$$Q(x) = (x-3)^3(x+2)(x^2+x+4)^2$$

where the last quadratic factor is irreducible (check by using the quadratic formula!), we have

$$\frac{P(x)}{Q(x)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{x+2} + \frac{Ex+F}{x^2+x+4} + \frac{Gx+H}{(x^2+x+4)^2}$$

(again this is in full generality and won't be asked in math 103. However you do need to know what happens if $Q(x)$ is a quadratic polynomial that cannot be factored (use completing the square and a substitution), has a repeated root (that uses substitution) or has distinct roots(this uses partial fractions as described here)).

(iii) Find A, B, C, D, E, F, G, H. (More on this later)

(iv) Integrate term by term.

Let's try this on our example $\int \frac{3x+1}{x^2+2x-3}\,dx$. Step one is already done. For step two, first we factor the denominator as $x^2+2x-3 = (x-1)(x+3)$. Now, we set up our problem as trying to find values of A and B so that

$$\frac{3x+1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)} = \frac{Ax+3A+Bx-B}{x^2+2x-3}$$

As the denominators are the same, we know that the numerators must be the same, that is

$$3x+1 = A(x+3)+B(x-1) = Ax+3A+Bx-B = (A+B)x+3A-B$$

Now there are two ways to find these values of A and B.

Method 1 - Plugging in values for x.

If we plug in $x=-3$, we see that $3(-3)+1 = A((-3)+3)+B(-3-1) = -4B$. Thus, $-8 = -4B$ so $B=2$. Next setting $x=1$, we see that $3(1)+1 = A(1+3)+B(1-1) = 4A$ and so $4=4A$ or $A=1$.

Method 2 - Comparing coefficients.

Comparing the coefficients of x gives $3 = A+B$ and comparing the constant term shows us that $1 = 3A-B$. adding these two equalities together gives $4=4A$ or $A=1$ and plugging this into $3=A+B$ shows that $B=2$.

Finishing up, we have

$$\begin{aligned}\int \frac{3x+1}{x^2+2x-3}\,dx &= \int \left(\frac{A}{x-1}+\frac{B}{x+3}\right)dx = \int \left(\frac{1}{x-1}+\frac{2}{x+3}\right)dx\\ &= \ln|x-1| + 2\ln|x+3| + C.\end{aligned}$$

Here are some examples

(i) $\int \frac{10}{x^2+3x-4}\,dx$

(ii) $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$

(iii) $\displaystyle\int \frac{x+1}{x^2-6x+5}\,dx$