Centroid

Goal: To compute a two dimensional centre of mass of an object with uniform density ρ.

We recall our one dimensional centre of mass formula given by

$$\bar{x} = \frac{1}{M}\int_a^b \rho x f(x)\, dx \qquad M = \int_a^b \rho f(x)\, dx = \rho A$$

where A is the area between the function $f(x)$ and the x-axis. In the two dimensional case, we simply repeat this computation in the x and y directions. This gives

$$\bar{x} = \frac{1}{M}\int_a^b \rho x f(x)\, dx = \frac{1}{\rho A}\int_a^b \rho x f(x)\, dx = \frac{1}{A}\int_a^b x f(x)\, dx$$
$$\bar{y} = \frac{1}{M}\int_c^d \rho y g(y)\, dy = \frac{1}{\rho A}\int_c^d \rho y g(y)\, dy = \frac{1}{A}\int_c^d y g(y)\, dy$$

where if $y = f(x)$ then $x = g(y)$ and so $f(g(y)) = y$, that is, $f(x)$ and $g(x)$ are inverse functions. There is an alternative definition for $\bar{y}$ given by

$$\bar{y} = \frac{1}{A}\int_a^b \frac{f(x)^2}{2}\, dx.$$

Think of this as the mass $\rho f(x)$ times half the distance from the x-axis integrated then normalized by the mass ρA. We call $(\bar{x}, \bar{y})$ the *centroid* or *the centre of mass of a two dimensional object.*

Let's do an example. Compute the centre of mass of a quarter circle of radius 3 found in the first quadrant with uniform density 3.