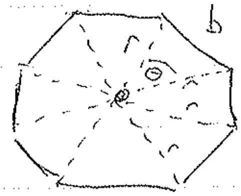
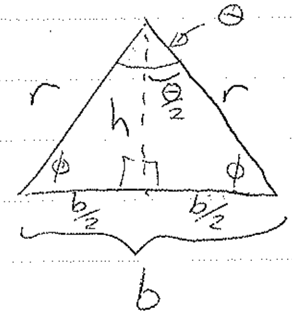
Carmen Bruni Jan 3/13

Piazza Access Code : ~~ubc 2012 w2 math103~~

Chapter 1: Area of a "n"-gon of sidelength "b"

$n=8$
octagon





$$A_{\text{Triangle}} = \frac{1}{2} b h \qquad n\theta = 2\pi \quad \text{so} \quad \theta = 2\pi/n$$

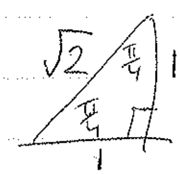
$$= \frac{1}{2} b \left(\frac{b}{2\tan(\theta/2)} \right) \qquad \tan(\theta/2) = \frac{b/2}{h} \quad \text{so} \quad h = \frac{b}{2\tan(\theta/2)}$$

$$= \frac{b^2}{4\tan(\theta/2)}$$

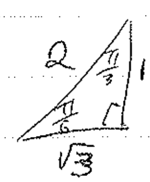
$$= \frac{b^2}{4\tan(\pi/n)}$$

$$A_{n\text{-gon}} = n A_{\text{Triangle}} = \frac{n b^2}{4\tan(\pi/n)} = \frac{nbh}{2}$$

Eg. Area of a unit square $= \dfrac{4(1)^2}{4\tan(\pi/4)} = \dfrac{1}{1} = 1$



Eg. Area of a unit hexagon $= \dfrac{36(1)^2}{24\tan(\pi/6)} = \dfrac{3}{2(\frac{1}{\sqrt{3}})} = \dfrac{3\sqrt{3}}{2}$



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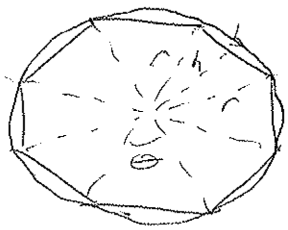
Area of a Circle:

Let $\pi = \dfrac{\text{Circumference of circle}}{\text{diameter of circle}} = \dfrac{C}{d} = \dfrac{C}{2r}$ r = radius.

Surround an "n"-gon by a circle.



As $n \to \infty$, the "n"-gon becomes the circle.

$$A_{n\text{-gon}} \xrightarrow{n \to \infty} A_{circle}.$$

$$A_{n\text{-gon}} = \frac{nbh}{2}$$

Perimeter of "n"-gon. $= nb$.
As $n \to \infty$, perimeter of the "n"-gon approaches the circumference of the circle $= 2\pi r$
As $n \to \infty$, $h \to r$

Combining gives

$$A_{n\text{-gon}} = \frac{(nb)(h)}{2} \xrightarrow{n \to \infty} \frac{(\cancel{2}\pi r)(r)}{\cancel{2}} = \pi r^2 = A_{circle}.$$

Tower of Hanoi

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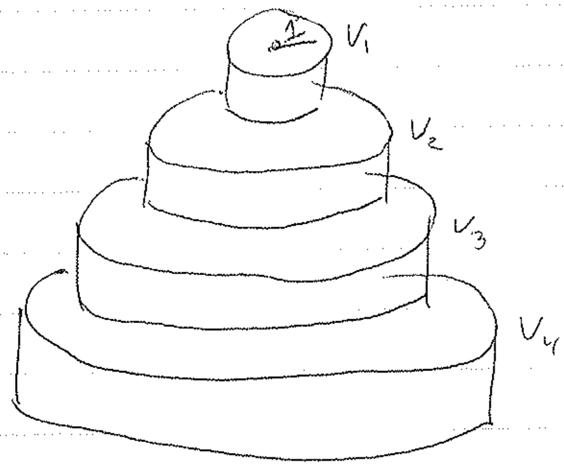

All cylinders have height 1. Radii will increase by 1 as we continue down the tower.

Reminder:

$V_{cyl.} = \pi r^2 h$

Q. Compute the volume of the 4-disc tower of Hanoi.

$$\begin{aligned} V_{Tower} &= V_1 + V_2 + V_3 + V_4 \\ &= \pi(1)^2(1) + \pi(2)^2(1) + \pi(3)^2(1) + \pi(4)^2(1) \\ &= \pi + 4\pi + 9\pi + 16\pi \\ &= 30\pi. \end{aligned}$$

Q: Compute the volume of the 100-disc tower of Hanoi.

$$\begin{aligned} V_{Tower} &= V_1 + V_2 + \cdots + V_{100} \\ &= \pi(1)^2(1) + \pi(2)^2(1) + \cdots + \pi(100)^2(1) \quad \text{... this seems hard.} \\ &= \sum_{i=1}^{100} \pi i^2 \\ &= \pi \sum_{i=1}^{100} i^2 \\ &= \pi\left(\frac{100(100+1)(2(100)+1)}{6}\right) \\ &= 338350\pi \end{aligned}$$

Sigma Notation

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A compact way to write long (or infinite) sums.

Let $\{a_i\}$ be a sequence of "n" real numbers. (n is an integer).

We write

$$\sum_{i=1}^{n} a_i \overset{\text{def.}}{=} a_1 + a_2 + \cdots + a_n$$

$n \leftarrow$ final index, $i=1 \leftarrow$ start index

Ex:

$$\sum_{i=2}^{5} (i+1) = ((2)+1) + ((3)+1) + ((4)+1) + ((5)+1)$$
$$= 3+4+5+6$$
$$= 18.$$

Ex: Write in sigma notation the sum of the first 10 natural numbers.

$$\sum_{i=1}^{10} i = (1)+(2)+(3)+ \cdots +(10) = 55.$$

Ex: Write in sigma notation

$$5+8+11+14+17+20 = \sum_{j=0}^{5} 5+3j$$

We can also add function values

Eg. $\sum_{k=1}^{3} f(k) = f(1)+f(2)+f(3).$

Infinite sums: $1-2+3-4+5-6+\cdots = \sum_{j=1}^{\infty} (-1)^{j+1} j = \sum_{j\geq 1} (-1)^{j+1} j$

Properties of Sigma Notation Jan 3/13

(1) Trivial sums $\sum_{j=n}^{n} a_j = a_n$ $\sum_{j=10}^{1} a_j \overset{\text{def.}}{=} 0$

c is an element of the real numbers

(2) Linearity Let $\{a_i\}$, $\{b_i\}$ be sequences and $c \in \mathbb{R}$. Then

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

(3) Independence of notation

$$\text{Ex: } \sum_{j=3}^{17} j \overset{\text{let } k=j-2}{=} \sum_{\underset{3-2}{k=1}}^{\overset{17-2}{15}} (k+2) = \sum_{j=1}^{15}(j+2)$$

(4) Adding ones Let $m, n \in \mathbb{Z}$ integers $m \le n$

$$\sum_{j=m}^{n} c = c\sum_{j=m}^{n} 1 = c(n-m+1)$$

$$\text{Ex: } \sum_{j=2}^{4} 2 = 2(4-2+1) = 2 \cdot 3 = 6$$

$\hookrightarrow = 2+2+2 = 6$

(5) Special sums.

(i) $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ (ii) $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ (iii) $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{j=1}^{n} j\right)^2$

Proof of (i)

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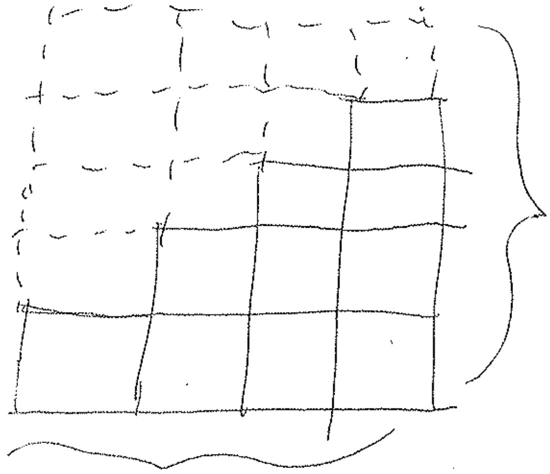

$n+1 = 4+1 = 5$

$4 = n$

$$\sum_{i=1}^{4} i = \frac{4 \cdot 5}{2} = 10$$

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① Question: Evaluate

$$\sum_{i=1}^{n} n \left(= \underbrace{n + n + \cdots + n}_{n \text{ copies of } n} = n^2 \right)$$

Answer.

(a) n

(b) n^2

(c) $\frac{n(n+1)}{2}$

(d) $n^{3/2}$

(e) Unsure.

(2)
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Compute $\sum_{i=13}^{30}(2+3i)$

<u>Ans</u>: $\sum_{i=13}^{30}(2+3i) = \sum_{i=13}^{30} 2 + \sum_{i=13}^{30} 3i$

$= 2\sum_{i=13}^{30}(1) + 3\sum_{i=13}^{30} i$

add # from 1 to 12 then subtracted them.

$= 2(30-13+1) + 3\left(\sum_{i=1}^{30} i - \sum_{i=1}^{12} i\right)$

$= 36 + 3\left(\frac{(30)(31)}{2} - \frac{(12)(13)}{2}\right)$

$= 36 + 3(465 - 78)$

$= 1197$

Geometric Series.

Let $r \neq 1$ be a real number (called the ratio) and $a \neq 0$ be a real number (the first term).

Then, a geometric series is of the form

$$S_N = \underset{a_0}{a} + \underset{a_1}{ar} + \underset{a_2}{ar^2} + \cdots + ar^N = \sum_{i=0}^{N} ar^i$$

~~Ex~~ NB: $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_N}{a_{N-1}}$

Ex: $S = 1 + 3 + 9 + 27 + 81 + 243 + 729.$

Then S is a geometric series

$$\frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \cdots = \frac{729}{243}$$

Ex: $S = 1 + 2 + 3 + 4$ is NOT a geometric series since

$$\frac{2}{1} \neq \frac{3}{2}.$$

Claim: $S_N = \frac{a(r^{N+1} - 1)}{r - 1} = \frac{a(1 - r^{N+1})}{1 - r}$

Pf: $S_N = a + ar + ar^2 + \cdots + ar^{N-1} + ar^N$ multiply by r

Subtract $rS_N = \quad ar + ar^2 + \cdots + ar^{N-1} + ar^N + ar^{N+1}$

$(1-r)S_N = a - ar^{N+1}$

Since $r \neq 1$, $1-r \neq 0$ and so

$$S_N = \frac{a - ar^{N+1}}{1-r} = \frac{a\left(1-r^{N+1}\right)}{1-r}. \qquad \boxed{\text{//}}$$

$$\sum_{i=0}^{N} ar^i = \frac{a\left(1-r^{N+1}\right)}{1-r}$$

Ex: Compute $\displaystyle\sum_{k=1}^{9} 3\cdot 2^k$

$a = \text{first term} = 3\cdot 2^1 = 6$
$r = \text{common ratio} = 2$
$N+1 = \#\text{ of terms} = 9-1+1 = 9$

Thus, $$\sum_{k=1}^{9} 3\cdot 2^k = \frac{6\left(1-(2)^9\right)}{1-2} = \frac{6-\cancel{3}6\cdot 2^9}{-1} = 3\cdot 2\cdot 2^9 - 6$$
$$= 3\cdot 2^{10} - 6$$
$$= 3\cdot 1024 - 6$$
$$= 3066$$

Alt. Method $$\sum_{k=1}^{9} 3\cdot 2^k = \sum_{k=0}^{8} 3\cdot 2^{k+1} = \sum_{k=0}^{\overset{N}{8}} \overset{a}{(3\cdot 2)}\,\overset{r}{2}^{k} = \frac{6\left(1-2^9\right)}{1-2} = 3066$$

$$r = \frac{12}{6} = 2 = \frac{24}{12} = \frac{48}{24} \ldots$$

Infinite Sums

Q: How do we add infinitely many numbers?

Idea: Add finitely many terms at the beginning (called the head) (a partial sum) then take the limit, That is

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$$

when the limit on the right exists

We call $S_N = \sum_{n=1}^{N} a_n$ the partial sum.

We say that an infinite sum <u>converges</u> if $\lim_{N \to \infty} S_N$ exists and is finite. and divergent otherwise.

Ex: $$\sum_{k=0}^{\infty} 2^k = \lim_{N \to \infty} \sum_{k=0}^{N} 2^k = \lim_{N \to \infty} \frac{(1)(1-2^{N+1})}{1-2} = \frac{1-\lim_{N \to \infty} 2^{N+1}}{-1}$$

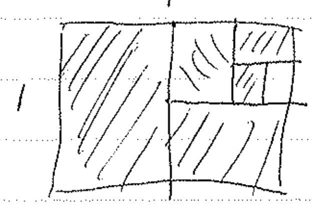
infinite

Hence this sum diverges.

$$\sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^k = \lim_{N \to \infty} \sum_{k=1}^{N} \left(\tfrac{1}{2}\right)^k = \lim_{N \to \infty} \frac{\left(\frac{1}{2}\right)\left(1-\left(\frac{1}{2}\right)^{N}\right)}{1-\frac{1}{2}} = 1.$$

N since I started at k=1

Hence the sum converges and $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$.



Area of square $= 1$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^k$$

In general, for geometric series

$$\sum_{n=0}^{\infty} ar^n \overset{\text{"="}}{=} \lim_{N\to\infty} \sum_{n=0}^{N} ar^n = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r}$$

$$= \frac{a(1-\lim_{N\to\infty} r^{N+1})}{1-r}$$

The sum converges when

$$\lim_{N\to\infty} r^{N+1} \text{ exists (and is finite).}$$

If $|r|>1$ the terms tend to $\pm$infinity
if $|r|=1$ then since $r\neq 1$, $r=-1$ and this limit does not exist.
if $|r|<1$ this converges!

<u>So</u> when $|r|<1$

$$\sum_{n=0}^{\infty} ar^n = \frac{a(\lim_{N\to\infty} r^{N+1}-1)}{r-1} = \boxed{\frac{a}{1-r}}$$

Ex: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{1}{2/3} = \frac{3}{2}$.

Thought experiment: Figure out why

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

diverges.

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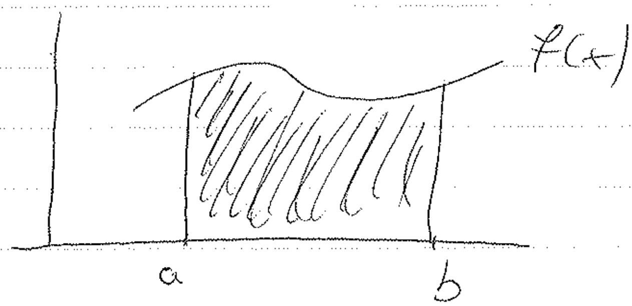
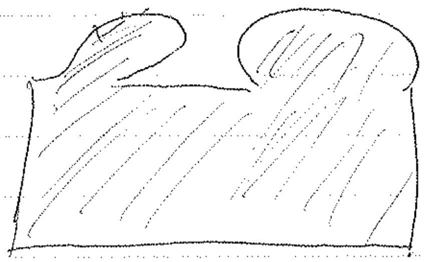
Chapter 2: Riemann Sums

Q: How do we compute the area of



Practical example: Putting carpet in a room



Idea: Use rectangles to approximate area!

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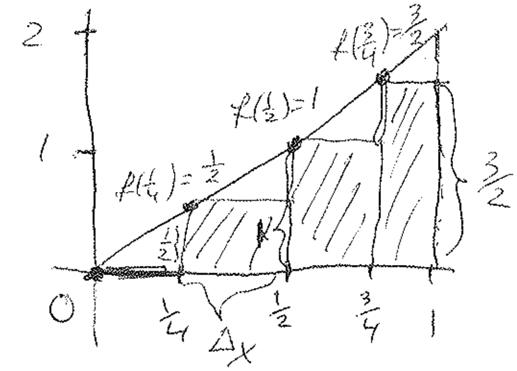
Simple Ex: $f(x) = 2x$ Want area under the curve between 0 & 1.



Approximate using left endpoints. and $n=4$ rectangles

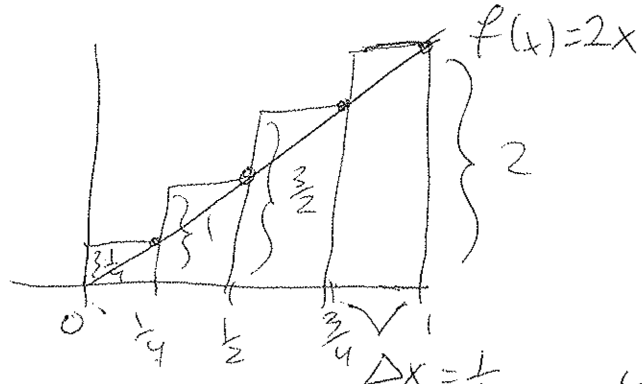
Actual Area: $\frac{1}{2}(2)(1) = 1$

Every rectangle has width $\Delta x = \frac{1}{4}$.

$$L_4 = \frac{1}{4} f(0) + \frac{1}{4} f(\tfrac{1}{4}) + \frac{1}{4} f(\tfrac{2}{4}) + \frac{1}{4} f(\tfrac{3}{4}) = \sum_{i=0}^{3} \frac{1}{4} f(\tfrac{i}{4}) = \sum_{i=1}^{4} \frac{1}{4} f(\tfrac{i-1}{4})$$

$$= \frac{1}{4}\left(0 + \frac{1}{2} + 1 + \frac{3}{2}\right)$$
$$= \frac{3}{4} < 1 \quad \text{(under estimate).}$$

Using right endpoint and 4 subintervals.



$$R_4 = \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + f(1)\right) = \sum_{i=1}^{4} \frac{1}{4} f(\tfrac{i}{4})$$

$$= \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{3}{2} + 2\right) = \frac{5}{4} > 1 \quad \text{(over estimate)}$$

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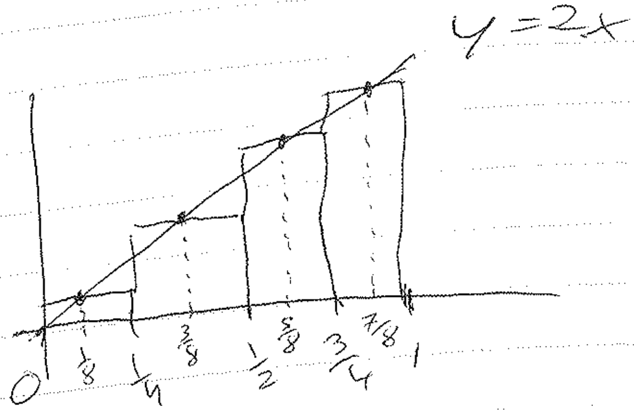
For "n" rectangles

$$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{2i}{n}\right) = \frac{2}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{n}{n}\cdot\frac{n+1}{n} = (1)\left(1+\frac{1}{n}\right) \xrightarrow{n\to\infty} 1$$

Use Midpoint Rule.



$\Delta x = \frac{1}{4}$

$$M_4 = \frac{1}{4}\left(f\left(\frac{\frac{1}{4}+0}{2}\right) + f\left(\frac{\frac{1}{2}+\frac{1}{4}}{2}\right) + f\left(\frac{\frac{3}{4}+\frac{1}{2}}{2}\right) + f\left(\frac{1+\frac{3}{4}}{2}\right)\right)$$

$$= \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$$

$$= \frac{1}{4}\sum_{i=1}^{n} f\left(\frac{2i-1}{8}\right) = \frac{1}{4}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)$$

$x_0 = 0$
$x_1 = \frac{1}{4}$
$x_2 = \frac{1}{2}$
$x_3 = \frac{3}{4}$
$x_4 = 1$

$$= \frac{1}{4}\left(\frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \frac{7}{4}\right)$$

$$= \underline{1}$$

(exact estimate).

General Setup

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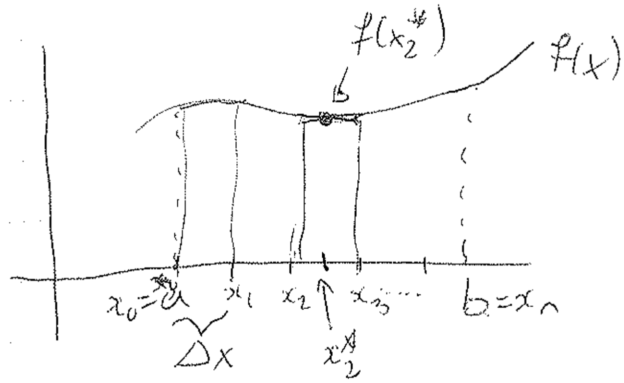
Let $f(x)$ be a function defined for $a \le x \le b$ $\quad a, b \in \mathbb{R}$.



Break into "n" rectangles

Δx = width of rectangle $= \frac{b-a}{n}$

$x_0 = a$ $\qquad x_2 = a + 2\Delta x$ $\qquad x_n = b$

$x_1 = a + \Delta x$ $\qquad \cdots \; x_i = a + i\Delta x$ $\qquad = a + n\Delta x$

Choose sample point $x_i^* \in [x_{i-1}, x_i]$

For example: Left endpoints $\quad x_i^* = x_{i-1}$
Right endpoints $\quad x_i^* = x_i$
Midpoint $\quad x_i^* = \frac{x_{i-1} + x_i}{2}$

Then, the area between the curve and the x-axis is approximately

$$\approx R_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

"R" for Riemann Sum.

and, Area under curve between a & b $= \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.

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NB: When $f(x)$ is continuous, this limit always exists.

Ex: $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

Left Riemann Sum

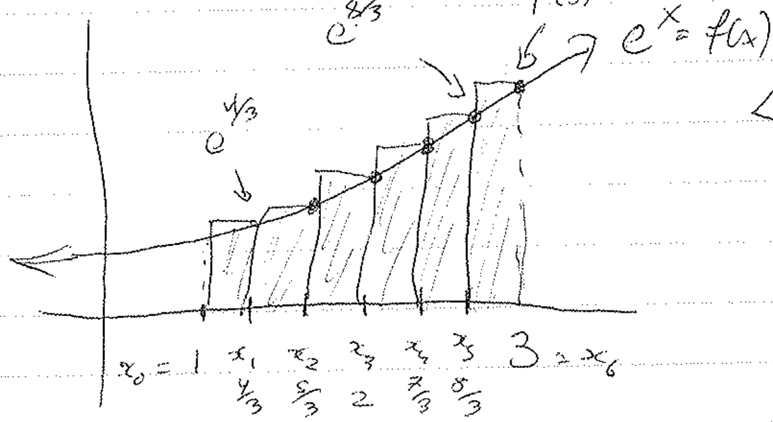
$= \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)\frac{b-a}{n}$

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ Right Riemann Sum

$= \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right)\frac{b-a}{n}$

$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$ Midpoint Riemann Sum

Ex: Use right endpoints & 6 subintervals to approximate the area under the curve $y = e^x$ from $x=1$ to $x=3$



$\Delta x = \frac{3-1}{6} = \frac{1}{3}$

$x_0 = a = 1$ $x_6 = b = 3$ $x_i = a + i\Delta x = 1 + \frac{i}{3}$.

$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \sum_{i=1}^{6} e^{1+i/3}\cdot\left(\frac{1}{3}\right)$

$= \frac{1}{3}\left(e^{4/3} + e^{5/3} + e^2 + e^{7/3} + e^{8/3} + e^3\right)$

≈ 20.42230852

NB Actual Area is $e^3 - e^1 \approx 17.36725509$

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Definite Integrals

Keeping the same notation (as in the Riemann sum setting) we have

$$\text{(signed) area between } f(x) \text{ and the x-axis} = \int_a^b f(x)\,dx \overset{\text{defined to be}}{:=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided the limit exists. (If it does, we say that $f(x)$ is integrable on $[a,b]$

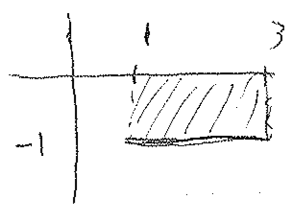
<u>Notes</u>:
- $\int$ is called an integral sign (\int in Piazza)
- $f(x)$ is called the integrand
- a & b are the limits of integration
- dx on its own has no meaning (for us)
- Integrals are limits of sums so $\int_a^b f(x)\,dx$ is a number (does not depend on x).

  <u>NB</u>: This number <u>CAN</u> be negative. (Integrals are signed areas).

  In fact. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(z)\,dz$.

  x, t, z are called dummy variables.
- The integral (amazingly) does <u>NOT</u> depend on choice of sample points x_i^*.

Eg. $\int_1^3 (-1)\,dx = -2$



Area of rectangle is $(2)(1) = 2$ <u>BUT</u> it is below the x-axis so its -2

<u>NB</u> f is integrable provided f is continuous <u>OR</u> has finitely many jump discontinuities.

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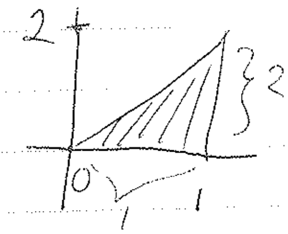
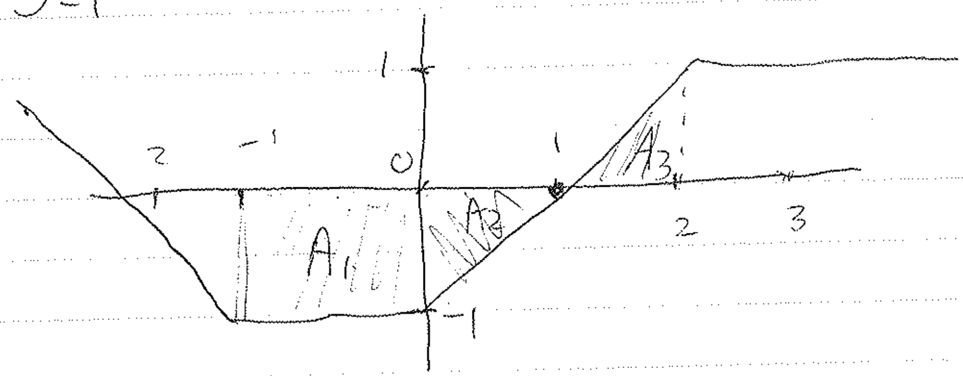
Ex: $\int_0^1 2x\,dx = 1$



Ex: Compute $\int_{-1}^{2} f(x)\,dx$ where $f(x)$ is defined by



$$\int_{-1}^{2} f(x)\,dx = -A_1 - A_2 + A_3$$
$$= -(1)(1) - \tfrac{1}{2}(1)(1) + \tfrac{1}{2}(1)(1)$$
$$= -1$$

Ex: Using right end point Riemann sums, compute $\int_0^3 3x^2\,dx$

Ans: $x_0 = a = 0 \qquad x_n = b = 3 \qquad f(x) = 3x^2$

$\Delta x = \frac{3-0}{n} = 3/n$

$x_i = a + i\Delta x = 0 + i(3/n) = 3i/n$

$$\int_0^3 3x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} 3\left(\frac{3i}{n}\right)^2 \frac{3}{n}$$
$$= \lim_{n\to\infty} \frac{81}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{81}{n^3} \frac{(n)(n+1)(2n+1)}{6}$$
$$= \lim_{n\to\infty} \frac{27}{2} \frac{n}{n}\cdot\frac{(n+1)}{n}\frac{(2n+1)}{n} = \lim_{n\to\infty} \frac{27}{2}(1)(1+\tfrac{1}{n})(2+\tfrac{1}{n})$$
$$= \frac{27}{2}\cdot 2 = \boxed{27}$$

Jan 10/13

Determine a region whose area is the Riemann Sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$$

Write as a definite integral.

Sol'n: Recall $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$

What should x_i be? guess.

$$x_i = a + i\Delta x = 5 + 2i/n$$

$$x_0 = a = 5.$$

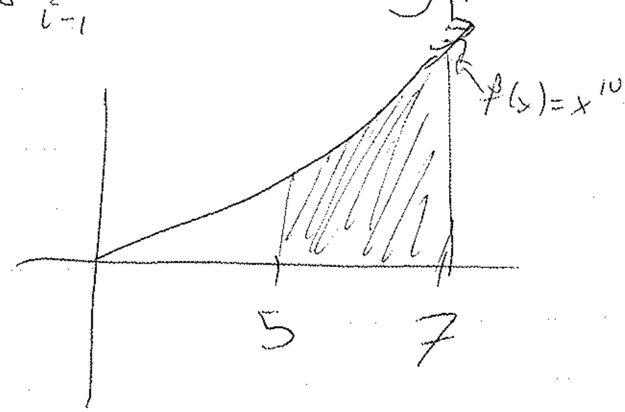
$$x_n = b = 7 = 5 + 2n/n$$

$$\Delta x = 2/n$$

$$f(x_i) = \left(5+\frac{2i}{n}\right)^{10} \qquad f(x) = x^{10}$$

$$= f(5 + 2i/n)$$

Hence $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_5^7 x^{10}\,dx$



Sample Problems

(1) Compute the following

(i) $$\sum_{n=0}^{10} \frac{2\cdot 3^{2n+1}}{15^n}$$

(ii) $$\sum_{n=3}^{10} \frac{2\cdot 3^{2n+1}}{15^n}$$

(iii) $$\sum_{n=3}^{\infty} \frac{2\cdot 3^{2n+1}}{15^n}$$

(2) Compute using high school geometry

$$\int_{-2}^{2} f(x)\,dx$$

Where $$f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x & \text{if } x > -1 \end{cases}$$

(3) Compute using high school geometry

$$\int_{-2}^{2} \left(\sqrt{4-x^2} + 1\right) dx$$

Hint: draw $f(x) = \sqrt{4-x^2}$ first (what shape is this?)
then shift by 1.

$$3^{2n+1} = 3^{2n}\cdot 3 = \cancel{9}(3^2)^n\cdot 3$$

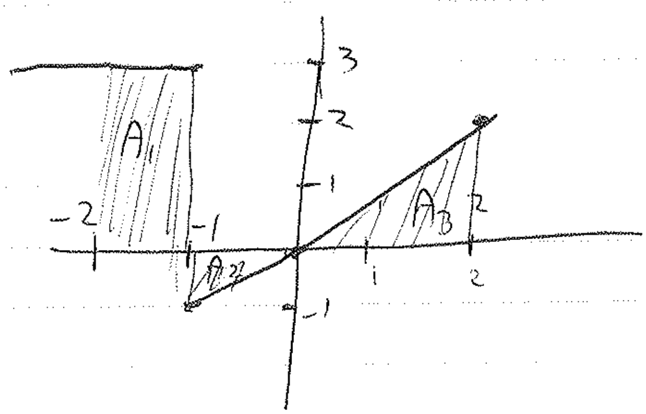
Sol'n

(1) (i) $$\sum_{n=0}^{10} \frac{2\cdot 3^{2n+1}}{15^n} = \sum_{n=0}^{10} 2\cdot 3\left(\frac{9}{15}\right)^n = \sum_{n=0}^{10} 6\left(\frac{3}{5}\right)^n = \frac{6\left(1-\left(3/5\right)^{10+1}\right)}{1-3/5}$$
$$= 15\left(1-\left(3/5\right)^{11}\right)$$

(ii) $$\sum_{n=3}^{10} \frac{2\cdot 3^{2n+1}}{15^n} = \sum_{n=3}^{10} 6\left(\frac{3}{5}\right)^n = \frac{6(3/5)^3\left(1-(3/5)^{10-3+1}\right)}{1-3/5} = \frac{81}{25}\left(1-(3/5)^8\right)$$

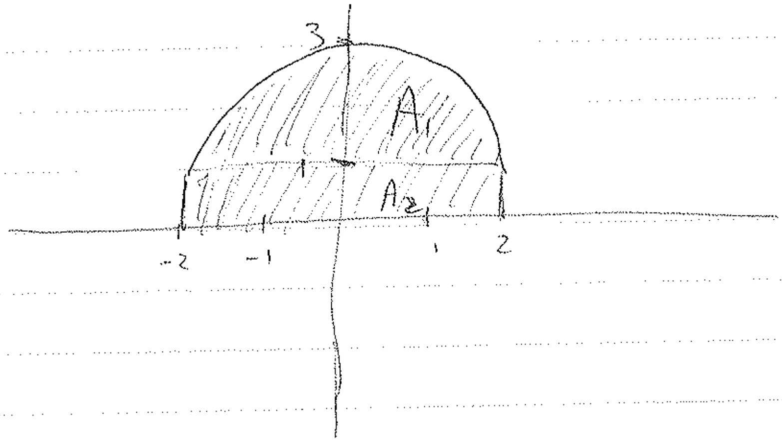
(iii) $$\sum_{n=3}^{\infty} \frac{2\cdot 3^{2n+1}}{15^n} = \sum_{n=3}^{\infty} 6\left(3/5\right)^n = \frac{6(3/5)^3}{1-3/5} = \left(6\cdot\frac{3^3}{5^3}\right)/(2/5)$$
$$= \frac{3^4}{5^2} = \frac{81}{25}$$

(2)



$$\int_{-2}^{2} f(x)\,dx = A_1 - A_2 + A_3 = (3)(1) - \tfrac{1}{2}(1)(1) + \tfrac{1}{2}(2)(2)$$
$$= 3 - \tfrac{1}{2} + 2$$
$$= 9/2$$

(3) $\int_{-2}^{2} \left(\sqrt{4-x^2}+1\right) dx$



$$\int_{-2}^{2} \left(\sqrt{4-x^2}+1\right) dx = A_1 + A_2$$
$$= \frac{1}{2}\pi(2)^2 + (4)(1)$$
$$= 2\pi + 4.$$

Announcements Jun 15/13

- Webwork 2 now open.
- Labdue Thursday (Friday) at beginning of class preferably.
- Midterm conflict?
  Let me know ASAP!!!
  M1: Monday Feb. 4 6-7pm
  M2: Thursday Mar. 7 6-7pm
- Review session 5-6:30 MATH 203 (I won't be there but we have the room).
- "E-mail Instructor" button.
- Starting Webwork late.

Last Time (DON'T COPY)

- Finished Riemann Sums
- The definite integral

$$\text{Area under Curve} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Today: The Fundamental Theorem of Calculus

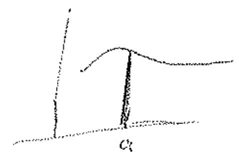
Properties of Definite Integrals.

Jun 15/13

Let $f(x)$ be integrable on $[a,b]$. and $c \in \mathbb{R}$.

(1) Trivial interval

$$\int_a^a f(x)\,dx = 0$$



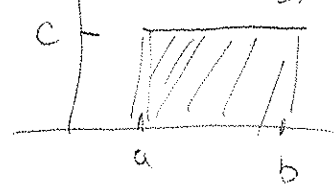
(2) Linearity

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

(3) Area of a rectangle.

$$\int_a^b c\,dx = c(b-a)$$



(4) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

ie $\Delta x = \frac{b-a}{n}$ becomes $\Delta x = \frac{a-b}{n}$

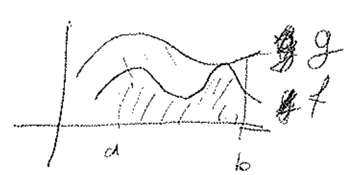
(5) Adding parts, If $c \in [a,b]$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(6) Positivity If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

(7) Boundedness: If $f(x) \leq g(x)$ for all $x \in [a,b]$ (and g is integrable) then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$



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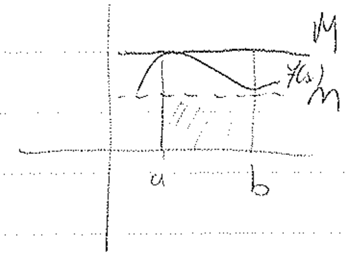
(8) Boundedness part 2 For $m, M \in \mathbb{R}$



If $m \leq f(x) \leq M$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx.$$

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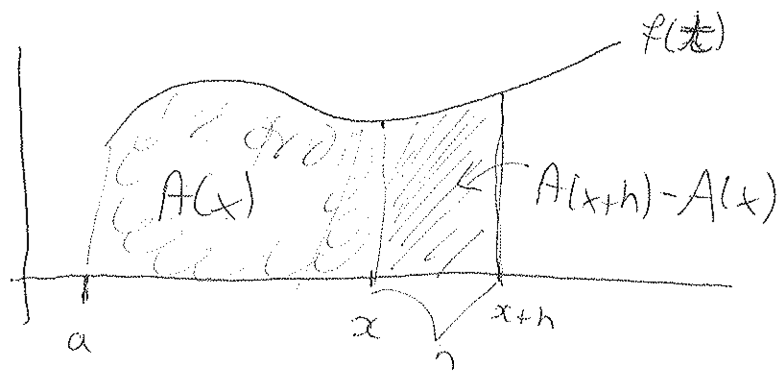
Chapter 3 Fundamental Theorem of Calculus

Change of paradigm: Think of integrals as functions.

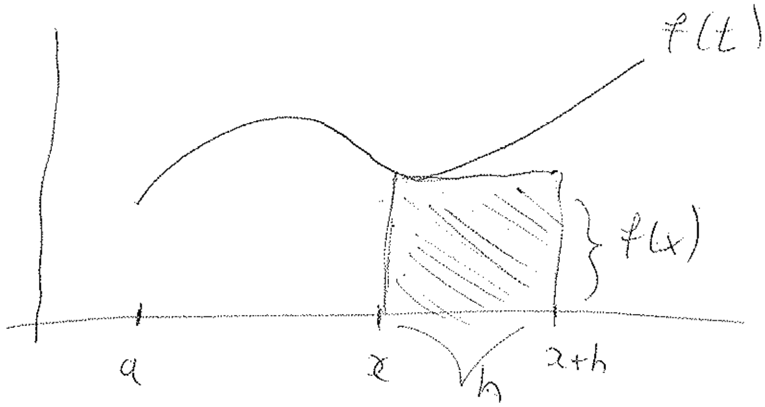
$$A(x) = \int_a^x f(t)\,dt$$

This is the area under the curve $f(t)$ from $t=a$ to $t=x$ where x can vary.

Let $\Delta x = h$ be a small number (NOT a height). Look at $A(x+h)$



$A(x+h) =$ curly area plus the lined area



So if h is REALLY small,

$$f(x)h \approx A(x+h) - A(x)$$

Hence $$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

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So as h gets really small

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \frac{dA}{dx} = A'(x)$$

Theorem: Fundamental Theorem of Calculus (Part I).

Let $f(x)$ be a bounded and continuous function on an interval $[a,b]$. Let

$$A(x) = \int_a^x f(t)\,dt$$

Then, for $x \in (a,b)$ (open interval from a to b), $A(x)$ is differentiable AND

$$\frac{dA}{dx} = f(x).$$

ie $A(x)$ is an antiderivative of $f(x)$.

Aside: Notice that $g_1(x) = x^2/2$ and $g_2(x) = x^2/2 + 1$ both differentiate to $f(x) = g_1'(x) = g_2'(x) = 2x$. In fact,

$$g_c = x^2/2 + C$$

are all antiderivatives of $f(x)$. In fact, any two anti-derivatives of a function must differ by a constant. (Proof: Use Mean Value Theorem).

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Theorem: Fundamental Theorem of Calculus (Part II)

Let $f(x)$ be a continuous function on $[a,b]$. Suppose $F(x)$ is <u>any</u> antiderivative of $f(x)$. Then for $x \in [a,b]$

$$A(x) = \int_a^x f(t)\,dt = F(t)\Big|_a^x := F(x) - F(a)$$

<u>PF:</u> We know $A(x)$ is an antiderivative of $f(x)$ (FTC Pt. I).
Thus

$$A(x) - F(x) = \int_a^x f(t)\,dt - F(x) = C \quad \text{for some constant } C.$$

Set $x = a$

$$C = \int_a^a f(t)\,dt - F(a) = -F(a)$$

Thus,

$$A(x) = \int_a^x f(t)\,dt = F(x) + C = F(x) - F(a) = F(t)\Big|_{t=a}^{x}$$

How to use FTC II

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Ex: $\int_0^1 2x\,dx = x^2\Big|_0^1 = (1)^2 - (0)^2 = 1$

Ex: $\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e^1$

Ex: $\int_0^4 (x^2 - 3x)\,dx = \int_0^4 x^2\,dx - 3\int_0^4 x\,dx = \frac{x^3}{3}\Big|_0^4 - 3\left(\frac{x^2}{2}\right)\Big|_0^4$

$= \frac{(4)^3}{3} - \frac{(0)^3}{3} - 3\left(\frac{(4)^2}{2} - \frac{(0)^2}{2}\right)$

$= \frac{64}{3} - \frac{48}{2} = -\frac{8}{3}$

OR $\int_0^4 (x^2 - 3x)\,dx = \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_0^4$

$= \frac{64}{3} - \frac{48}{2}$

$= -8/3$

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Antiderivative Table

	Function $f(x)$	Antiderivative $F(x)$	
$\hat{C} \in \mathbb{R}$	$\hat{C}$	$\hat{C}x$	$+C$
$m \in \mathbb{R}, m \neq -1$	x^m	$\frac{1}{m+1} x^{m+1}$	$+C$
$0 \neq b \in \mathbb{R}$	$\cos(bx)$	$\frac{1}{b} \sin(bx)$	$+C$
	$\sin(bx)$	$\frac{-1}{b} \cos(bx)$	$+C$
	$\sec^2(bx)$	$\frac{1}{b} \tan(bx)$	$+C$
	e^{bx}	$\frac{1}{b} e^{bx}$	$+C$
	$\frac{1}{x}$	$\ln\lvert x\rvert$	$+C$
	$\frac{1}{1+x^2}$	$\arctan(x)$	$+C$
	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$	$+C$

Aside

Jan 15/13

$$\int \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \to \int 0$$

$$\boxed{\arcsin(x) + \arccos(x) = C}$$

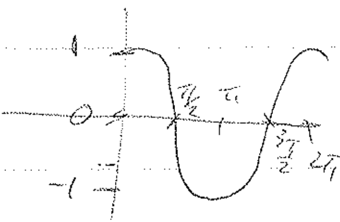
$$\arcsin(0) + \arccos(0) = C$$

$$0 + \pi/2 = C$$

$$\therefore \arcsin(x) + \arccos(x) = \pi/2$$

Sample Problem   NOT NECESSARY   Jan 15/13

(1) $$\int_{-2\pi}^{\pi} \sin(2x)\,dx = \left(\frac{-\cos(2x)}{2} + 1\right)\Bigg|_{-2\pi}^{\pi}$$



$$= \left(-\frac{\cos(2\pi)}{2} + 1\right) - \left(-\frac{\cos(-4\pi)}{2} + 1\right)$$

$$= \left(\frac{-1}{2} + 1\right) - \left(-\frac{1}{2} + 1\right)$$

$$= 0$$

(2) $$\int_{-2}^{0} e^{2x+1}\,dx = e\int_{-2}^{0} e^{2x}\,dx = e\left(\frac{e^{2x}}{2}\Bigg|_{-2}^{0}\right)$$

$$= e\left(\frac{e^{2(0)}}{2} - \frac{e^{2(-2)}}{2}\right)$$

$$= e\left(\frac{1}{2} - \frac{e^{-4}}{2}\right)$$

$$= \frac{e}{2} - \frac{e^{-3}}{2} = \frac{e}{2} - \frac{1}{2e^3}$$

(3) $$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Bigg|_{-1}^{1} = \frac{(1)^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} \boxed{= 0}$$

(4) $$\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Bigg|_{-1}^{1} = \frac{(1)^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

(5) $$2\int_{0}^{1} x^2\,dx = 2\left(\frac{x^3}{3}\Bigg|_{0}^{1}\right) = 2\left(\frac{(1)^3}{3} - \frac{(0)^3}{3}\right) = \frac{2}{3}$$ equal.

Symmetric Functions

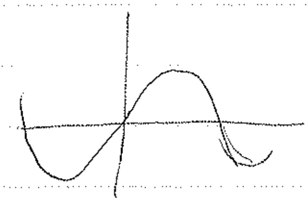
Jan 15/13

Def'n: We say that a function $f(x)$ is
- even if $f(-x) = f(x)$ (reflection about y-axis)
- odd if $f(-x) = -f(x)$ (rotation by $180°$ about origin).

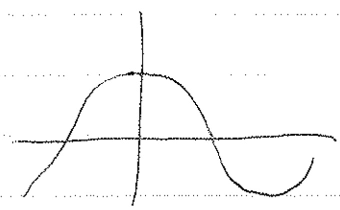
Ex: $f(x) = x^2 + x^4$ is even. Since
$$f(-x) = (-x)^2 + (-x)^4 = x^2 + x^4 = f(x)$$

$f(x) = x^3 + x^5$ is odd since
$$f(-x) = (-x)^3 + (-x)^5 = -x^3 - x^5 = -(x^3 + x^5) = -f(x).$$

$f(x) = \sin(x)$ is odd since
$$f(-x) = \sin(-x) = -\sin(x) = -f(x)$$



$f(x) = \cos(x)$ is even since
$$f(-x) = \cos(-x) = \cos(x) = f(x)$$



Correction

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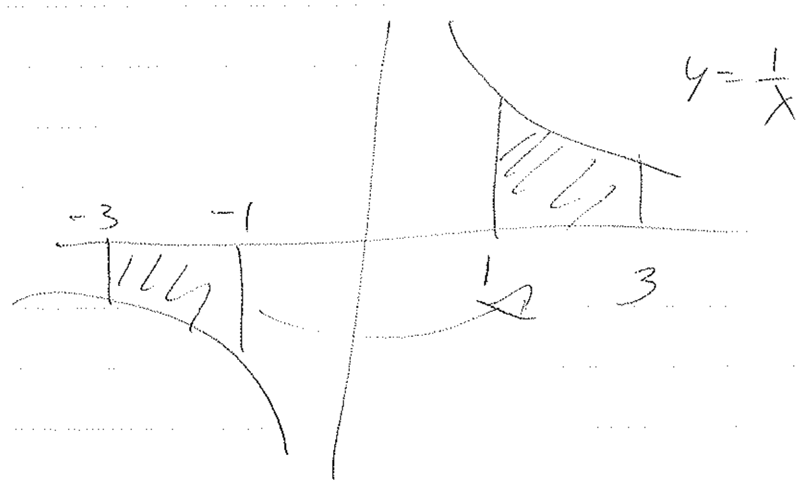
$$\int_{-3}^{-1} \frac{1}{x}\,dx = \ln(x)\Big|_{-3}^{-1} = \ln(-1) - \ln(-3)$$

BAD cannot take log of negative.

Solution: Change antiderivative of $\frac{1}{x}$ to be $\ln|x| + C$.

$$\int_{-3}^{-1} \frac{1}{x}\,dx = \ln|x|\Big|_{-3}^{-1} = \ln|-1| - \ln|-3|$$
$$= \ln(1) - \ln(3)$$
$$= 0 - \ln(3)$$
$$= \ln(3^{-1}) = \ln(1/3)$$

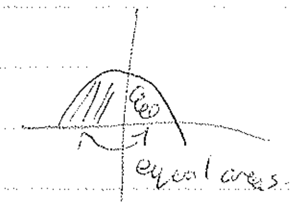


Back to Even & Odd Function.   Jun 17/13

Proposition: Suppose that $f(x)$ is integrable on $[-a, a]$ where $a \in \mathbb{R}$. Then

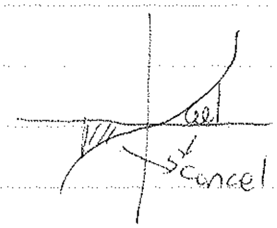
(1) If $f(x)$ is even, we have

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



(2) If $f(x)$ is odd, we have

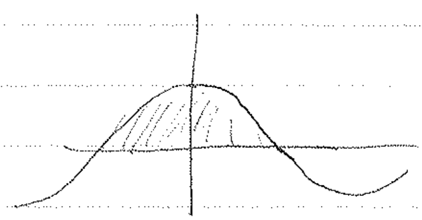
$$\int_{-a}^{a} f(x)\,dx = 0$$



Ex. $\displaystyle\int_{-3}^{3} x^{11}\cos(x^3)\,dx$.

Let $f(x) = x^{11}\cos(x^3)$ Then

$$\begin{aligned} f(-x) &= (-x)^{11}\cos((-x)^3) \\ &= -x^{11}\cos(-x^3) \\ &= -x^{11}\cos(x^3) \\ &= -f(x) \end{aligned}$$



Hence $f(x)$ is odd. Since $[-3, 3]$ is symmetric about the origin, we have $\displaystyle\int_{-3}^{3} x^{11}\cos(x^3)\,dx = 0$

NB: Note that some functions ie. $y = x^3 + x^2$ are neither even nor odd.

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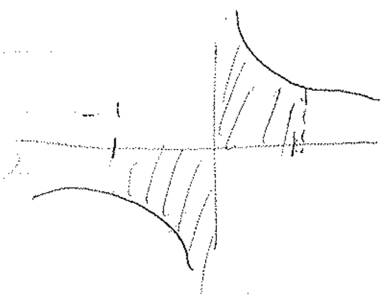
CAUTION

Evaluate $\int_{-1}^{1} \frac{dx}{x^3}$



$y = \frac{1}{x^3}$

Well $f(x) = \frac{1}{x^3}$ is odd so this is 0 right? Maybe...

$$\int_{-1}^{1} \frac{dx}{x^3} = \int_{-1}^{1} x^{-3}\,dx \overset{\text{FTC II}}{=} \frac{-x^{-2}}{2}\Big|_{-1}^{1} = \frac{-(1)^{-2}}{2} - \left(\frac{-(-1)^{2}}{2}\right)$$
$$= -\frac{1}{2} + \frac{1}{2} = 0. \quad \text{right?}$$

<u>NO</u>! Big problem: $f(x) = \frac{1}{x^3}$ is <u>NOT</u> continuous at $0 \in [-1, 1]$

<u>So</u> $f(x)$ is not necessarily integrable on $[-1, 1]$ Moreover, FTC II does not apply since $f(x)$ is not continuous.

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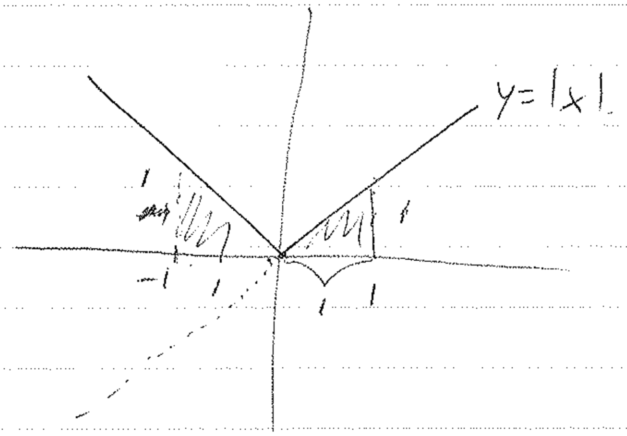
Ex: $\int_{-1}^{1} x\,dx = 0$

$\int_{-1}^{1} |x|\,dx$

Pictorially



So $\int_{-1}^{1} |x|\,dx = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1.$

OR Note for $x \geq 0$, $|x| = x$

$x \leq 0$ $|x| = -x$ NOTE: Since x is negative "$-x$" is positive

So
$$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx$$
$$= \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$$
$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$$
$$= -\left(\frac{(0)^2}{2} - \frac{(-1)^2}{2}\right) + \left(\frac{(1)^2}{2} - \frac{(0)^2}{2}\right)$$
$$= 1.$$

KEY IDEA: Figure out when $f(x)$ is positive and negative, break into pieces and integrate $\int_a^b |f(x)|\,dx$

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Ex: $\int_1^4 \frac{x^2+3x}{\sqrt{x}}\,dx = \int_1^4 \frac{x^2}{\sqrt{x}}\,dx + \int_1^4 \frac{3x}{\sqrt{x}}\,dx$

$= \int_1^4 x^{3/2}\,dx + \int_1^4 3\sqrt{x}\,dx$

$= \frac{2x^{5/2}}{5}\Big|_1^4 + \frac{\cancel{3}\cdot 2x^{3/2}}{\cancel{3}}\Big|_1^4$

$= \frac{2\sqrt{4}^5}{5} - \frac{2\sqrt{1}^5}{5} + 2\sqrt{4}^3 - 2\sqrt{1}^3$

$= \frac{2\cdot 2^5}{5} - \frac{2}{5} + 2\cdot 2^3 - 2$

$= \frac{64}{5} - \frac{2}{5} + 16 - 2$

$= \frac{132}{5}$

Sample problems

1. $\int_{-1}^{1} \sin^2(x) \cos^2(x) \tan(x)\, dx$

2. $\int_{1}^{4} \frac{6x^6 + \sqrt{x} + 1}{x}\, dx$

3. $\int_{-2}^{3} |x^2 - 1|\, dx$ (Don't simplify the final numeric answer)

4. $\int_{0}^{1} 2x e^{x^2}\, dx$

(For those with experience with integration, try:)

5. $\int_{3}^{4} (3x^2+1) \cos(x^3+x)\, dx$

6. $\int_{1}^{e} \frac{dx}{x(x+1)}$

7. $\int_{-1}^{1} \arccos(x)\, dx$

8. If $F(x) = \int_{1}^{x^2} \sin(t)\, dt$, find $F'(x)$.

Answers

1. $\displaystyle\int_{-1}^{1} \sin^2(x)\cos(x)\tan(x)\,dx = \int_{-1}^{1} \sin^2(x)\cos(x)\frac{\sin(x)}{\cos(x)}\,dx = \int_{-1}^{1} \sin^3(x)\,dx$

Now $\sin^3(-x) = (\sin(-x))^3 = (-\sin(x))^3 = -\sin^3(x)$

So $\sin^3(x)$ is an odd function. Since the interval is symmetric about 0, we have

$$\int_{-1}^{1} \sin^2(x)\cos(x)\tan(x)\,dx = 0$$

2. $\displaystyle\int_1^4 \frac{6x^6+\sqrt{x}+1}{x}\,dx = \int_1^4 \left(\frac{6x^6}{x} + \frac{\sqrt{x}}{x} + \frac{1}{x}\right)dx = \int_1^4 \left(6x^5 + x^{-\frac{1}{2}} + \frac{1}{x}\right)dx$

$$= x^6 + 2\sqrt{x} + \ln|x| \Big|_1^4$$

$$= 4^6 + 2\sqrt{4} + \ln(4) - \left((1)^6 + 2\sqrt{1} + \ln(1)\right)$$

$$= 2^{12} + 4 + \ln(4) - \cancel{1} - \cancel{2} - \underbrace{\ln(1)}_{=0}$$

$$= 4096 + 1 + \ln(4)$$

$$= 4097 + \ln(4)$$

3. We need to find when x^2-1 is below the x-axis, that is, when $x^2-1<0$ that is, when $x^2<1 \Rightarrow -1<x<1$ so

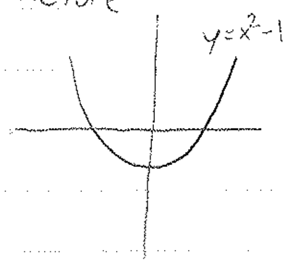
$$\int_{-2}^{3} |x^2-1|\,dx = \int_{-2}^{-1} \underbrace{(x^2-1)}_{\text{positive}}\,dx + \int_{-1}^{1} \underbrace{-(x^2-1)}_{\text{negative flips this to positive}}\,dx + \int_{1}^{3} \underbrace{(x^2-1)}_{\text{positive}}\,dx$$

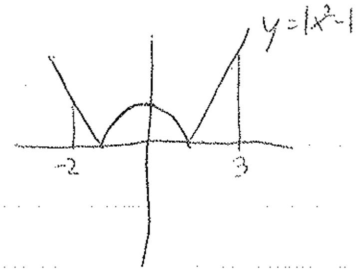
$$= \left(\frac{x^3}{3} - x\right)\Bigg|_{-2}^{-1} - \left(\frac{x^3}{3} - x\right)\Bigg|_{-1}^{1} + \left(\frac{x^3}{3} - x\right)\Bigg|_{1}^{3}$$

$$\int_{-2}^{3} |x^2-1|\,dx = \frac{(-1)^3}{3} - (-1) - \left(\frac{(-2)^3}{3} - (-2)\right) - \left[\frac{(1)^3}{3} - (1) - \left(\frac{(-1)^3}{3} - (-1)\right)\right]$$
$$+ \left(\frac{(3)^3}{3} - 3 - \left(\frac{(1)^3}{3} - 1\right)\right)$$

$= \cdots$ (more steps)
$= 28/3$

Picture





4. $\int_0^1 2xe^{x^2}\,dx$

Notice $\frac{d}{dx} x^2 = 2x$ So $\frac{d}{dx} e^{x^2} = 2xe^{x^2}$ Thus

$$\int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e^{(1)^2} - e^{(0)^2} = e-1.$$

Answers for harder problems

5. $\int_3^4 (3x^2+1)\cos(x^3+x)\,dx = \sin(68) - \sin(30)$

6. $\int_2^e \frac{dx}{x(x-1)} = \ln(2) - \ln(e+1) + 1$

7. $\int_{-1}^1 \arccos(x)\,dx = \pi$

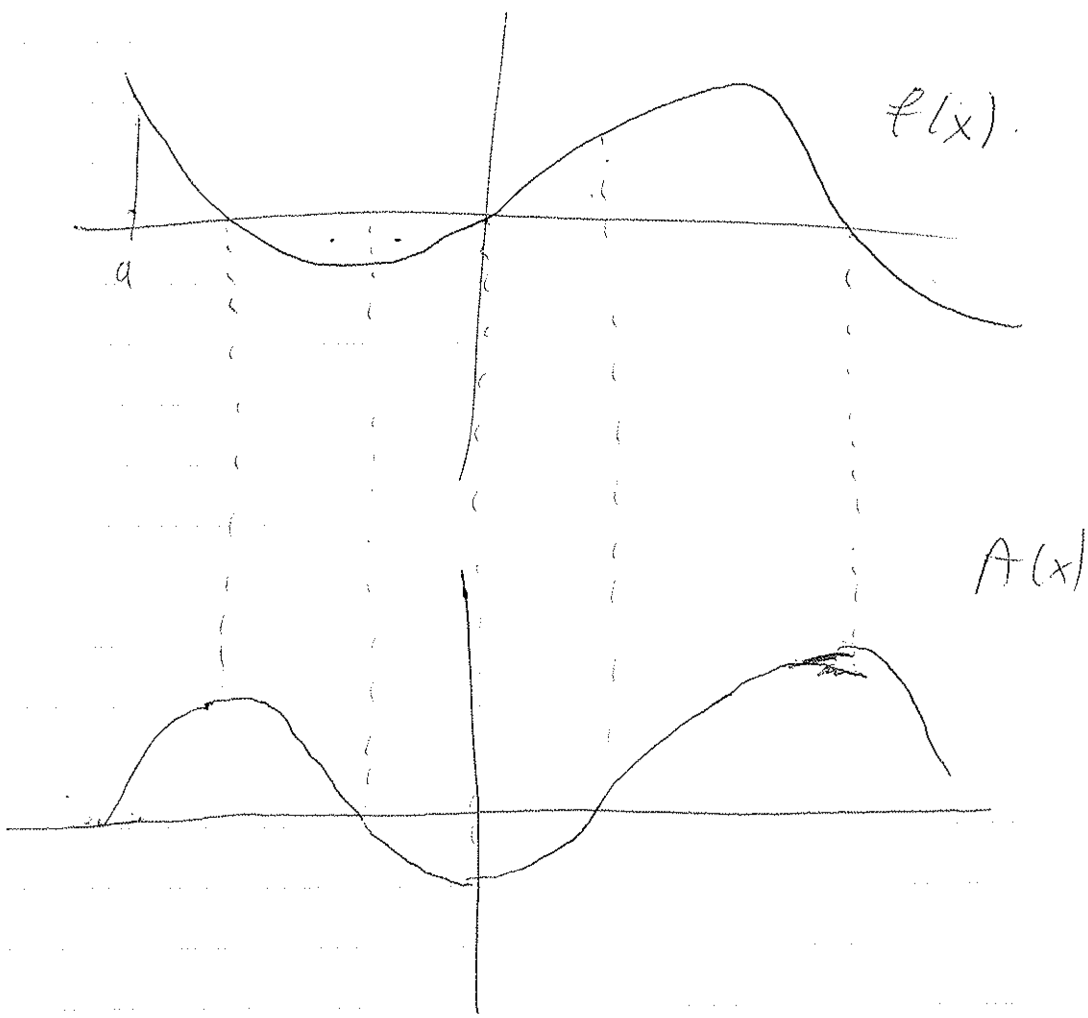
8. $F(x) = \int_1^{x^2} \sin(t)\,dt$ $\quad F'(x) = \sin(x^2)\cdot 2x$

Sketching Integrals

Jan 17/13

Given $f(x)$ as shown, draw $A(x) = \int_a^x f(t)\,dt$

Idea: Same as given $g'(x)$ draw $g(x)$



Area Between Two Curves — Jan 17/13

Find the area between $y=x^2$ & $y=x^3$

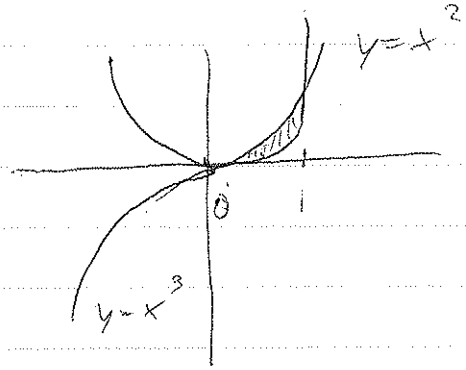
<u>Sol'n</u>: where do the curves intersect?

This happens when $x^3 = x^2$

$$x^3 - x^2 = 0$$

$$x^2(x-1) = 0$$

$$x=0 \quad \text{OR} \quad x=1$$

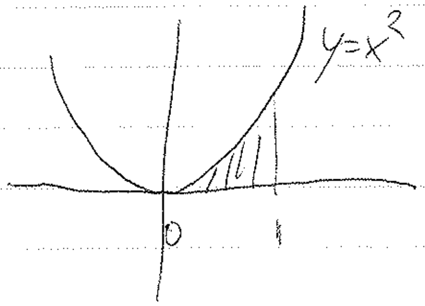


Note if $0<x<1$

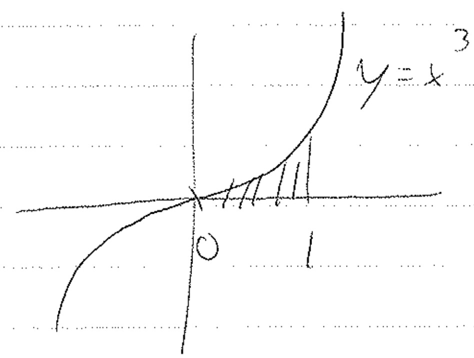
$0<x^3<x^2$

So x^2 is bigger on $[0,1]$.

To computed the shaded area,



Subtract



Hence, Area between $y=x^2$ & $y=x^3$ $\displaystyle\int_0^1 (x^2-x^3)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

<u>NB</u> In general, the area between two curves $f(x)$ & $g(x)$ from a to b is given by

$$\int_a^b |f(x) - g(x)|\,dx$$

§4 Applications of Integration.

Jan 22/13

Net Change Theorem: The integral of a rate of change is the net change. ie.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

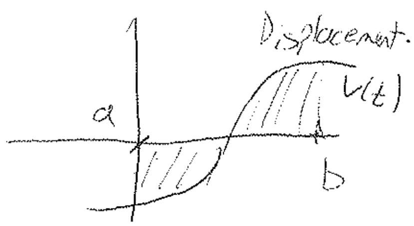
Ex: Let $a(t)$ be the acceleration of a particle.
Let $v(t)$ be the velocity of a particle
Let $x(t)$ be the position of a particle.

Since $\frac{dv}{dt} = a(t)$

$$\int_a^b a(t)\,dt = v(b) - v(a)$$

net change in velocity.

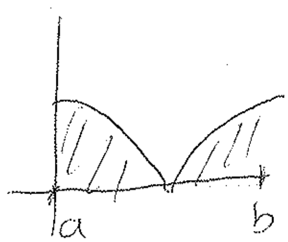
Since $\frac{dx}{dt} = v(t)$



$$\int_a^b v(t)\,dt = x(b) - x(a)$$

displacement.
(net change of position)

How do we get total distance?



$$\int_a^b |v(t)|\,dt = \text{total distance.}$$

Sample Problem. Jan 22/13

Find $x(t)$ given $x(0)$, $v(0)$ and $a(t) = g = 9.8\ m/s^2$ = constant.

$$v(T) - v(0) = \int_0^T a(t)\,dt = \int_0^T g\,dt = gt\Big|_0^T = gT - g(0) = gT.$$

Hence $v(T) = gT + v(0)$

Now, integrate velocity

$$x(t) - x(0) = \int_0^t v(T)\,dT = \int_0^t (gT + v(0))\,dT$$

$$= \left(\frac{gT^2}{2} + v(0)T\right)\Big|_0^t$$

$$= \frac{gt^2}{2} + v(0)t$$

Thus, $x(t) = x(0) + v(0)t + \frac{g}{2}t^2$

$$\int_0^t t\,dt$$

Jan 22/13

Better Approximation of a falling object. Assume $v(0)=0$, $K \in \mathbb{R}$ is a constant. Suppose

$$a(t) = \frac{dv}{dt} = g - Kv$$

drag force.

<u>Question</u>: Compute $v(T)$.

$$\boxed{\text{Recall: If } \frac{dy}{dt} = -Ky \text{ then } y(t) = y(0)e^{-Kt}}$$

<u>So</u> $\frac{da}{dt} = -K\frac{dv}{dt} = -K\,a(t)$ Thus $a(t) = a(0)e^{-Kt}$

Since $a(0) = g - Kv(0) = g - K(0) = g$ Hence $a(t) = ge^{-Kt}$

<u>So</u> $v(T) - v(0) = \int_0^T a(t)\,dt = \int_0^T ge^{-Kt}\,dt$

$$= g\left(\frac{e^{-Kt}}{-K}\Big|_0^T\right)$$

$$= g\left(\frac{e^{-KT}}{-K} - \frac{\overset{1}{e^{-K(0)}}}{-K}\right)$$

$$= g\left(\frac{1}{K} - \frac{1}{Ke^{KT}}\right)$$

$$= \frac{g}{K}\left(1 - \frac{1}{e^{KT}}\right)$$

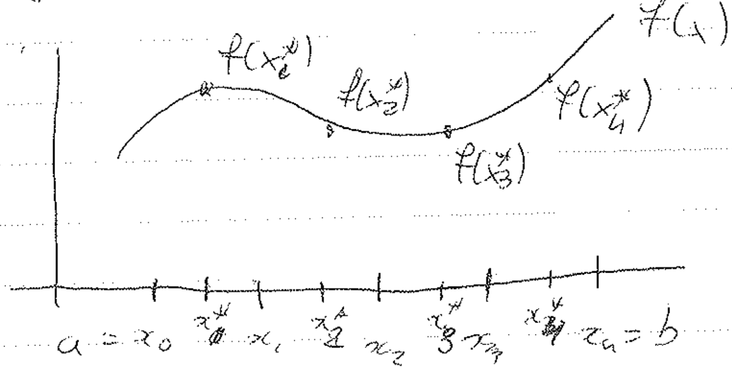
$$\boxed{\frac{d}{dt}e^{-Kt} = -Ke^{-Kt}}$$

<u>So</u> $v(T) = \frac{g}{K}\left(1 - \frac{1}{e^{KT}}\right) + \underbrace{v(0)}_{=0} = \frac{g}{K}\left(1 - \frac{1}{e^{KT}}\right)$

As $T \to \infty$, $v(T) \to \frac{g}{K}$ = terminal velocity

Average Value

Jan 22/13



$n = 4$

$x_i^* \in [x_{i-1}, x_i]$

What is the average of the $f(x_i^*)$?

$$\frac{f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)}{4}$$

To generalize: Average value of $f \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i^*)$

Looks like a Riemann sum but missing Δx

$\Delta x = \frac{b-a}{n} \Rightarrow \frac{\Delta x}{b-a} = \frac{1}{n}$ Then take limits.

Def'n: The average value of f on $[a,b]$ is

$$f_{av} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^*) = \lim_{n\to\infty} \frac{\Delta x}{b-a} \sum_{i=1}^{n} f(x_i^*)$$

$$= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

$$\boxed{f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

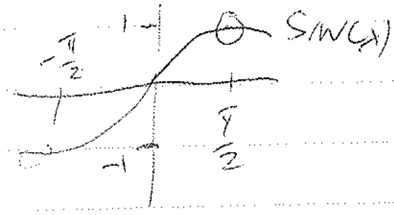
Jan 22/13

Ex: Find the average value of $f(x) = \cos(x)$ on $[\underset{a}{-\frac{\pi}{2}}, \underset{b}{\frac{\pi}{2}}]$.

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{(\frac{\pi}{2})-(-\frac{\pi}{2})}\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$$

$$= \frac{1}{\pi} \sin(x)\Big|_{-\pi/2}^{\pi/2}$$



$$= \frac{1}{\pi}\left(\sin(\tfrac{\pi}{2}) - \sin(-\tfrac{\pi}{2})\right)$$

$$= \frac{1}{\pi}\left(1 - (-1)\right)$$

$$= \frac{2}{\pi}.$$

Jan 22/13

Problems:

up to fall equinox.

(1) Compute the average length of a day between spring & summer.
Assume that length of a day is measured by

$$f(t) = 12 + 4\sin\left(\frac{2\pi t}{366}\right)$$

So this function is cyclic of period 366 days $\approx$ # of days in a year. The function also begins at the spring equinox when $t=0$.

(2) The temperature of a cup of juice is

$$T(t) = 20(1 - e^{-t}) \text{ degrees Celsius}$$

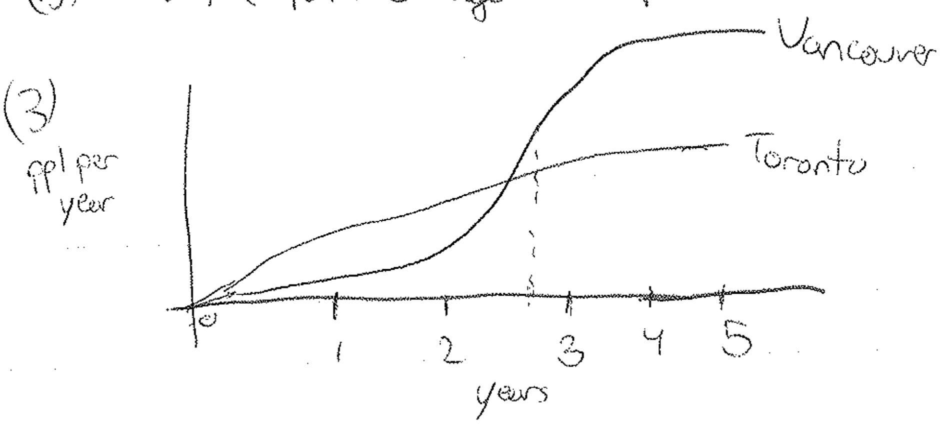
And the rate of change of a cup of coffee is

$$f(t) = 50e^{-2t} \text{ degrees celsius per minute.}$$

(t is in minutes)

(a) Find the change in temperature of Juice between $t=1$ and $t=5$
(b) Find the total change in temperature of the coffee between $t=1$ & $t=5$.

(3)



Above is shown the population growth rates of Toronto & Vancouver. Which has a higher population in year 1? Year 2.6? Year 4?

Jan 22/13

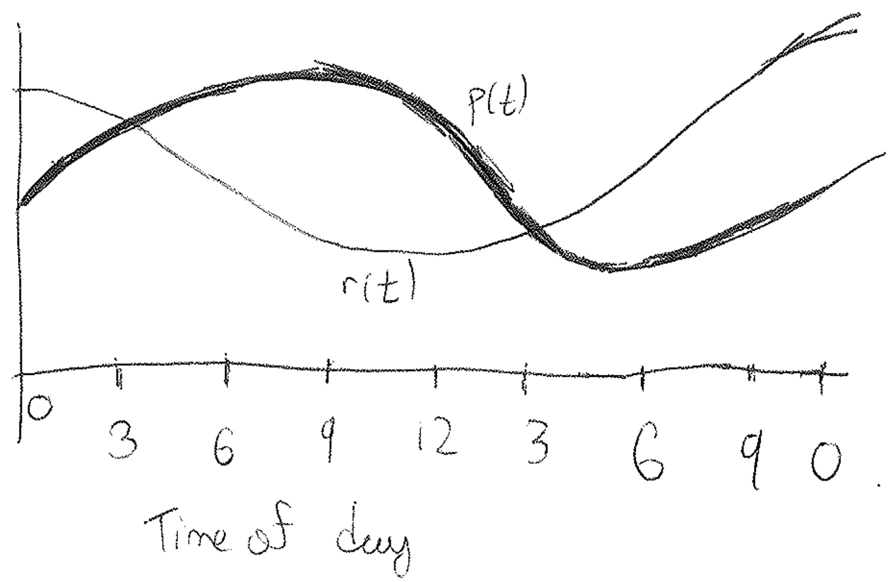
(4) A baby grows at a rate of

$$g(t) = 20 - 1.1t \quad 0 \le t \le 20$$

in cm.

How tall is the child at 20 years?

(5)



periodic function based on circadian rhythm.

Suppose that a certain hormone is being secreted into the blood stream. Let $p(t)$ be its production and $r(t)$ be its removal rate.

1. When is the production rate maximal?
2. When is the removal rate minimal?
3. At what time is the hormone level in the blood highest?
4. At what time is the hormone level in the blood lowest?

At 3 pm.

At 3 am (the graph continues periodically forever).

Jan 22/13

(1) # of days in ~~a~~ spring & summer: $\frac{366}{2} = 183.$

$$f_{av} = \frac{1}{183-0}\int_0^{183} 12 + 4\sin\left(\frac{2\pi t}{366}\right) dt$$

$$= \frac{1}{183}\left(12t\Big|_0^{183} + \frac{4\left(-\cos\left(\frac{2\pi t}{366}\right)\right)}{\frac{2\pi}{366}\ \frac{\pi}{183}}\Bigg|_0^{183}\right)$$

$$= \frac{1}{183}\left(12(183) - \frac{4\cdot 183}{\pi}\left(\cos(\pi) - \cos(0)\right)\right)$$

$$= \frac{1}{\cancel{183}}\left(12(\cancel{183}) - 4\frac{\cancel{183}}{\pi}\left(-1-(1)\right)\right)$$

$$= 12 + \frac{8}{\pi} \approx 14.7$$

(2) (a) $T(5) - T(1) = 20(1-e^{-5}) - 20(1-e^{-1})$

$$= \frac{20}{e} - \frac{20}{e^5}$$

(b) $\displaystyle\int_1^5 50e^{-2t}\,dt = 50\left(\frac{e^{-2t}}{-2}\Big|_1^5\right)$

$$= -25\left(e^{-10} - e^{-2}\right)$$

$$= \frac{25}{e^2} - \frac{25}{e^{10}}$$

(3) Winner Year 1 Toront Year 2.6 Toronto Year 4 Vancouver.

(we are comparing AREAS).

Solution. Jan 22/13

(4) $\int_0^{20} (20 - 1.1t)\,dt = \left(20t - \frac{1.1t^2}{2}\right)\Big|_0^{20}$

$= 20 \cdot 20 - \frac{1.1}{2}(20)^2$

$= 400 - 1.1 \cdot 200$

$= 400 - 220$

$= 180$ cm.

Jan 24/13

Revisit The Hormone Level

Suppose that you produce a hormone at a rate of

$$p(t) = 1 + \sin(\omega t)$$

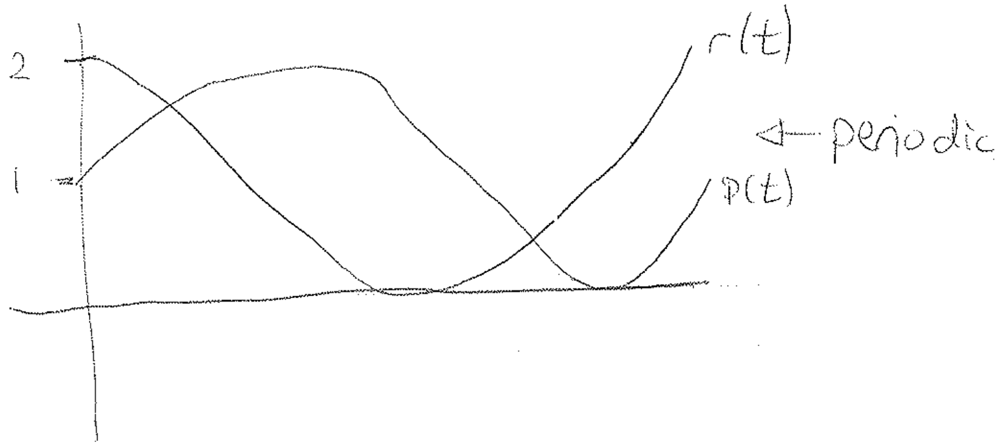
and you use a hormone at a rate of

$$r(t) = 1 + \cos(\omega t).$$ t in hours

where $\omega = \frac{\pi}{12}$

Compute the hormone level in your body when it is maximal given that when your hormone level is minimal, it is 0.

Sol'n



Minimal at first intersection point. Maximal hormone level at second intersection point.

To find intersection points,

$$p(t) = r(t)$$
$$1 + \sin(\omega t) = 1 + \cos(\omega t)$$
$$\sin(\omega t) = \cos(\omega t)$$

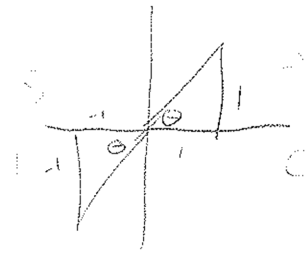
So either $\cos(\omega t) = 0$ (which means $\sin(\omega t)$ is not zero so this cannot occur) or we can divide by $\cos(\omega t)$ to get

$$\frac{\sin(\omega t)}{\cos(\omega t)} = 1$$

$$\tan(\omega t) = 1.$$

Jun 24/13

So since



$\tan(\theta) = 1$ when $\theta = \frac{\pi}{4}$ OR $\frac{5\pi}{4}$

Thus, $\omega t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ or since $\omega = \frac{\pi}{12}$, $t = 3$ or 15 (ie 3am or 3pm).

Hence, we want

$$\int_3^{15} (p(t) - r(t))\,dt = \int_3^{15} (1 + \sin(\omega t) - (1 + \cos(\omega t)))\,dt$$

$$= \int_3^{15} (\sin(\omega t) - \cos(\omega t))\,dt$$

$$= -\frac{\cos(\omega t)}{\omega} - \frac{\sin(\omega t)}{\omega} \Big|_3^{15}$$

$\boxed{\omega = \frac{\pi}{12}}$

$$= \frac{1}{\omega}\left(-\cos\left(\frac{\pi}{12}\cdot 15\right) - \sin\left(\frac{\pi}{12}\cdot 15\right)\right) - \left(-\cos\left(\frac{\pi}{12}\cdot 3\right) - \sin\left(\frac{\pi}{12}\cdot 3\right)\right)$$

$$= \frac{1}{\frac{\pi}{12}}\left(-\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\right) - \left(-\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{12}{\pi}\left(\frac{2}{\sqrt{2}} - \left(\frac{-2}{\sqrt{2}}\right)\right)$$

$$= \frac{12}{\pi}\left(\frac{4}{\sqrt{2}}\right) = \frac{24\sqrt{2}}{\pi}$$

Jan 24/13

$I(t)$ = # of infections

$$= 100000\left(\cos\left(\frac{t}{6}\pi\right) + \cos\left(\frac{t\pi}{120}\right) + 2\right)$$

Want to minimize the average value of $I(t)$ between some month x and $x+60$ ($60 = 5 \times 12 = $ # of months). That is, minimize

$$F(x) = \frac{1}{x+60-x}\int_{x}^{x+60} I(t)\,dt.$$

To optimize, differentiate & set to 0.

$$\frac{d}{dx}F(x) = \frac{1}{60}\frac{d}{dx}\int_{x}^{x+60} I(t)\,dt$$

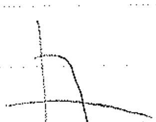
$$= \frac{1}{60}\frac{d}{dx}\left(\int_{x}^{0} I(t)\,dt + \int_{0}^{x+60} I(t)\,dt\right)$$

$$= \frac{1}{60}\frac{d}{dx}\left(-\int_{0}^{x} I(t)\,dt + \int_{0}^{x+60} I(t)\,dt\right)$$

FTC part 1 $\rightarrow$ $= \frac{1}{60}\left(-I(x) + I(x+60)\frac{d}{dx}(x+60)\right)$ Chain rule.

set to 0

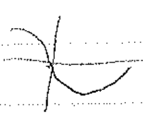
$$0 = \frac{1}{60}\left(-100000\left(\cos\left(\frac{\pi}{6}x\right) + \cos\left(\frac{\pi x}{120}\right) + 2\right) + 100000\left(\cos\left(\frac{\pi}{6}(x+60)\right) + \cos\left(\frac{\pi}{120}(x+60)\right) + 2\right)\right)$$



$$0 = -\cos\left(\frac{\pi}{6}x\right) - \cos\left(\frac{\pi}{120}x\right) + \cos\left(10\pi + \frac{\pi}{6}x\right) + \cos\left(\frac{\pi}{2} + \frac{\pi x}{120}\right)$$

shift by $\frac{\pi}{2}$

$$0 = -\cos\left(\frac{\pi}{6}x\right) - \cos\left(\frac{\pi}{120}x\right) + \cos\left(\frac{\pi}{6}x\right) - \sin\left(\frac{\pi}{120}x\right)$$



$$0 = -\cos\left(\frac{\pi}{120}x\right) - \sin\left(\frac{\pi}{120}x\right)$$

$$\sin\left(\frac{\pi}{120}x\right) = -\cos\left(\frac{\pi}{120}x\right)$$

$$\tan\left(\frac{\pi}{120}x\right) = -1 \quad \left(\text{since } \cos\left(\frac{\pi}{120}x\right) \neq 0\right).$$

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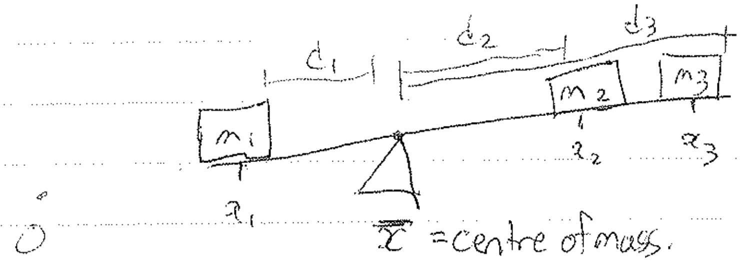
Hence $\frac{\pi x}{120} = \frac{3\pi}{4}$ OR $\frac{7\pi}{4}$.

<u>So</u> $x = 90$ <u>OR</u> $x = 210$.

Notice $F(90) \approx 109968 \leftarrow$ local min.

$F(210) \approx 290031$

§5 Applications Density & Mass. Jan 24/13



$\bar{x}$ = centre of mass.

Q: what is the total mass of the boxes? $\sum_{i=1}^{3} m_i = M$.
(in general, $M = \sum_{i=1}^{n} m_i$)

Balanced when

$$m_1 d_1 = m_2 d_2 + m_3 d_3$$

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) + m_3(x_3 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} + m_3\bar{x} = m_1 x_1 + m_2 x_2 + m_3 x_3$$

$$\bar{x}(m_1 + m_2 + m_3) = \sum_{i=1}^{3} m_i x_i$$

$$\Rightarrow \quad \bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{M}$$

in general $\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{M} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}}$

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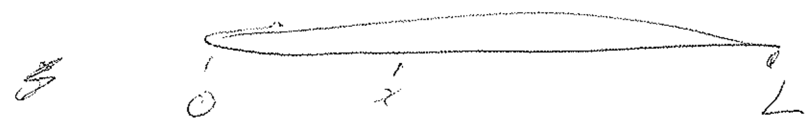
Now, consider a javelin



We want to know the mass & centre of mass. Suppose that the density of the javelin is given by $\rho(x)$

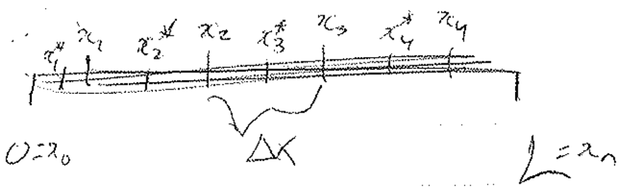
If $\rho(x)$ was constant, then

$$\text{mass} = \text{density} \times \cancel{\text{length}}\ \text{volume}.$$

(OR if "one dimensional") mass = density × length.

However, our density might vary. Over a different point, we might have a different density.

To account for this, break up into pieces



So on each piece $\quad m_i = \rho(x_i^*)\Delta x.$

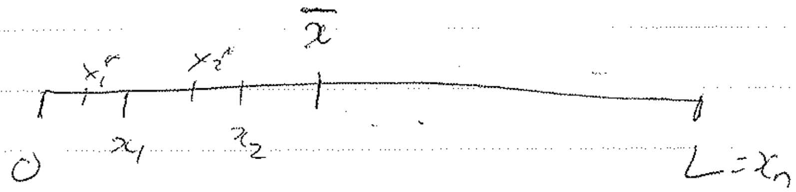
Sum over all pieces. $\quad \sum_{i=1}^{n} m_i = \sum_{i=1} \rho(x_i^*)\Delta x$

Take limit. $\quad \lim_{n\to\infty} \sum_{i=1}^{n} m_i = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$

$$\boxed{M = \int_0^L \rho(x)\,dx}$$

Jan 24/13

For centre of mass-



$$\text{Centre of mass} \approx \frac{\sum_{i=1}^{n} \overbrace{x_i^*}^{\text{distance}} \overbrace{\rho(x_i^*)\Delta x}^{\text{mass}}}{\sum_{i=1}^{n} \rho(x_i^*)\Delta x}$$ ← This is discrete mass

Thus,
$$\bar{x} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \rho(x_i^*)\Delta x}{\lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x}$$

$$\bar{x} = \frac{\int_0^L x\rho(x)\,dx}{\int_0^L \rho(x)\,dx}$$ ← this is mass

$$\boxed{\bar{x} = \frac{1}{M}\int_0^L x\rho(x)\,dx}$$

NB: Think of mass as a function. Define a cumulative function of mass as

$$M(x) = \int_0^x \rho(x)\,dx$$

Example of Density

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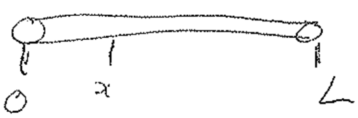
Ex: Let $a \in \mathbb{R}$. Suppose a long thin bar of length L is made of material whose density varies along the length of the bar. Let x be the distance from one end and suppose its density is given by

$$\rho(x) = ax \qquad 0 \leq x \leq L.$$



(a) Find the total mass of the bar
(b) Find the average mass density along the bar.
(c) Find the centre of mass of the bar.
(d) Where must you cut the bar to get two pieces of equal mass.

Sol'n: (a) $M = \int_0^L ax\,dx = \frac{ax^2}{2}\Big|_0^L = \frac{aL^2}{2} - 0 = \frac{aL^2}{2}$.

(b) NB This is the average density - Use average value formula.

$$\bar{\rho} = \frac{1}{b-a}\int_a^b \rho(x)\,dx = \frac{1}{L}\int_0^L \rho(x)\,dx = \frac{1}{L}\int_0^L ax\,dx = \frac{1}{L}\cdot\frac{ax^2}{2}\Big|_0^L = \frac{aL^2}{2L}$$
$$= aL/2.$$

(c) $$\bar{x} = \frac{\int_0^L x\rho(x)\,dx}{M} = \frac{\int_0^L ax^2\,dx}{aL^2/2} = \frac{2}{aL^2}\,\frac{ax^3}{3}\Big|_0^L = \frac{2aL^3}{3aL^2} - 0$$
$$= \frac{2}{3}L$$

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(d) Want to know ~~the~~ x value when

$$\frac{M}{2} = \int_0^x \rho(t)\,dt$$

$$= \int_0^x at\,dt$$

$$= \frac{at^2}{2}\Big|_0^x$$

$$= \frac{ax^2}{2}$$

Isolating for x gives $x = \sqrt{\frac{M}{a}} = \sqrt{\frac{aL^2}{2a}} = \frac{L}{\sqrt{2}} = \frac{L\sqrt{2}}{2}$

Area b/w curves.

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Ex: Find the area of the region enclosed by

$f(x) = x+2 \qquad g(x) = x^2-4.$

Find pts of intersection

$$f(x) = g(x)$$
$$x+2 = x^2-4$$
$$0 = x^2-x-6 = (x-3)(x+2)$$
so $x = -2, 3.$

Since $f(0) > g(0)$ $f(x) > g(x)$ on $[-2,3]$.

$$A = \int_{-2}^{3} (f(x)-g(x))\,dx = \int_{-2}^{3} (x+2-x^2+4)\,dx$$
$$= \int_{-2}^{3} (-x^2+x+6)\,dx$$
$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right)\Big|_{-2}^{3}$$
$$= -\frac{(3)^3}{3} + \frac{(3)^2}{2} + 6(3) - \left(\frac{-(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right)$$
$$= \frac{125}{6}.$$

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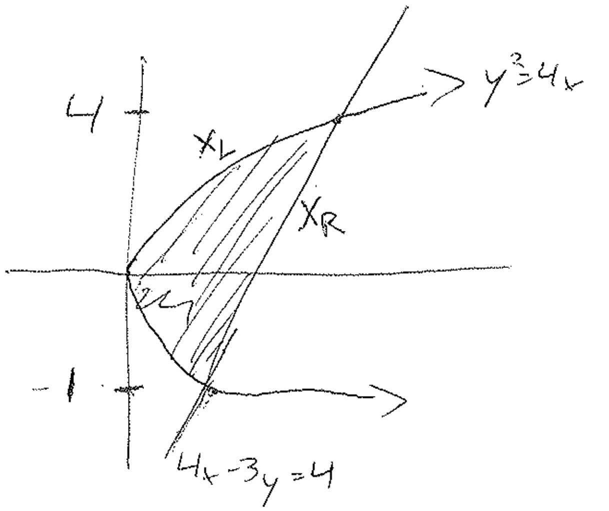
Ex: Find the area between $y^2=4x$ and $4x-3y=4$.



Find points of intersection.

$$y^2=4x$$
$$4x-3y=4$$
$$y^2-3y-4=0$$
$$(y-4)(y+1)=0$$
$$y=-1,4$$

Isolating for x gives

$$x=\frac{y^2}{4} \qquad x=\frac{4+3y}{4}=1+\frac{3}{4}y.$$

$$\int_{-1}^{4}(x_R-x_L)\,dy = \int_{-1}^{4}\left(\left(1+\frac{3}{4}y\right)-\frac{y^2}{4}\right)dy$$
$$=\left(y+\frac{3}{8}y^2-\frac{y^3}{12}\right)\Big|_{-1}^{4}$$
$$=\dots$$
$$=\frac{125}{24}.$$

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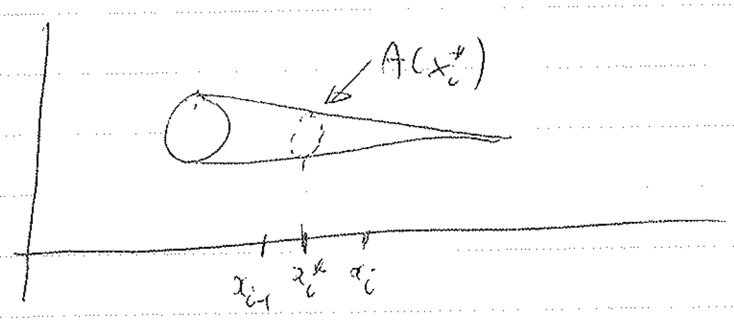
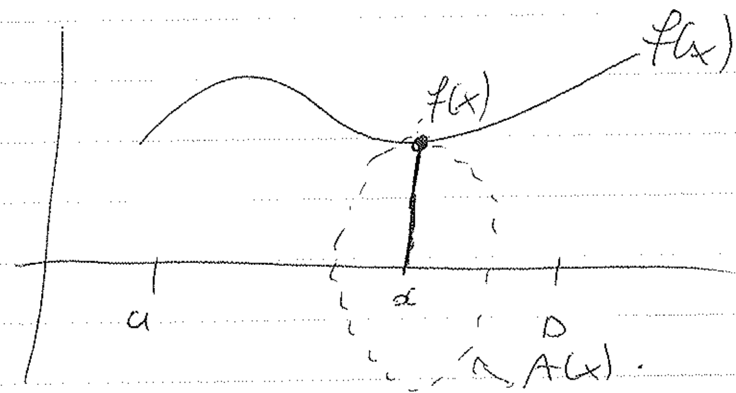
Volumes of Revolution

Def'n: Let S be a solid between $x=a$ and $x=b$. Let $A(x)$ be the cross-sectional area of S in the plane through x and perpendicular to the x-axis. Assume that $A(x)$ is continuous. Then,

$$\text{Vol}(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$



Solids of Revolution



$$A(x) = \pi r^2 = \pi f(x)^2$$

So $\quad \text{Volume} = \displaystyle\int_a^b \pi f(x)^2\,dx.$

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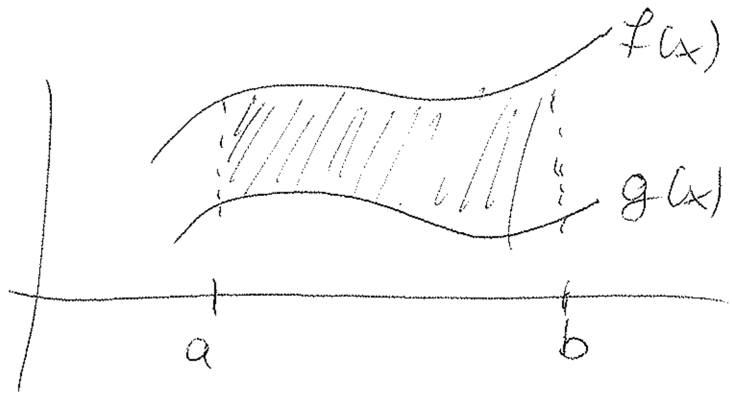
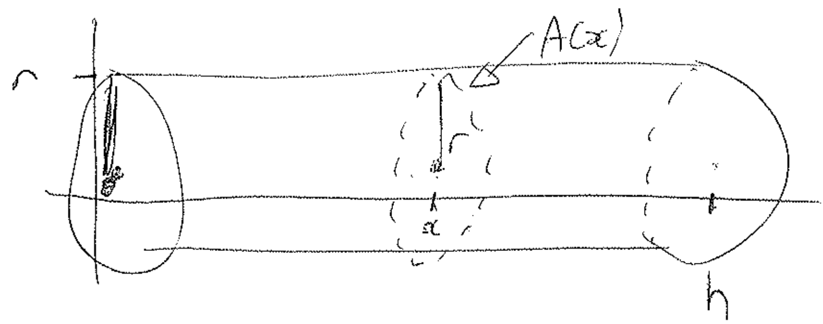
Between two curves.



Rotate shaded region & compute its volume.

Volume of revolved region $= \int_a^b \left(\pi f(x)^2 - \pi g(x)^2\right) dx$.

Ex: Compute the volume of a cylindrical banana.



$$Vol = \int_0^h A(x)\,dx = \int_0^h \pi r^2\,dx$$

NOTICE that the cross sectional area does NOT depend on x.

$$= \pi r^2 x \Big|_0^h$$

$$= \pi r^2 h.$$

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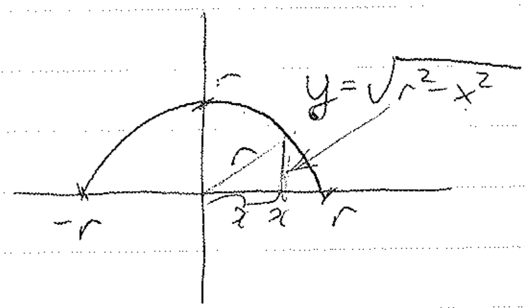
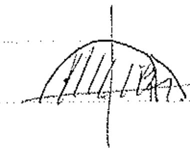
Volume of a sphere. $x^2+y^2=r^2$ r fixed.





$$Vol = \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx.$$

$$= 2\int_0^r \pi(r^2-x^2)\,dx$$ (since function is even & interval is symmetric).

$$= 2\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r$$

$$= 2\pi\left(r^3-\frac{r^3}{3}\right)$$

$$= \frac{4\pi}{3}r^3.$$

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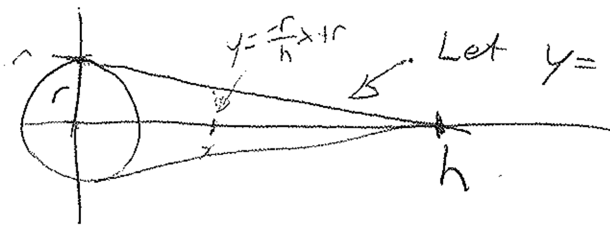
Volume of a cone of radius r and height h.



Let $y=mx+b$ be the top line.

When $x=0$, $y=r$ $\quad r=m(0)+b=b$.
when $y=0$, $x=h$ $\quad 0=mh+r \Rightarrow m=\frac{-r}{h}$.

So the top line has equation $y=\frac{-r}{h}x+r$.

Using solids of revolution,

$$\text{Volume}=\int_0^h \pi\left(-\frac{r}{h}x+r\right)^2 dx$$

$$=\int_0^h \pi r^2\left(\frac{x^2}{h^2}-\frac{2x}{h}+1\right)dx$$

$$=\pi r^2\left(\frac{x^3}{3h^2}-\frac{x^2}{h}+x\right)\Big|_0^h$$

$$=\pi r^2\left(\frac{h^3}{3h^2}-\frac{h^2}{h}+h-0\right)$$

$$=\pi r^2\left(\frac{h}{3}-h+h\right)$$

$$=\frac{\pi r^2 h}{3}$$

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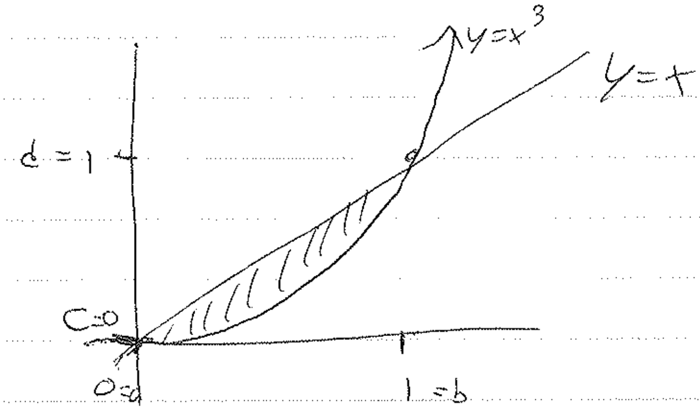
Ex: Find the volume of the solid of revolution between the ~~axes~~ curves $y=x$ and $y=x^3$ in the first quadrant about the x-axis.



Points of intersection are $(0,0)$ and $(1,1)$ [and $(-1,-1)$].

$$Vol = \int_0^1 \left(\pi x^2 - \pi (x^3)^2\right) dx = \pi \int_0^1 (x^2 - x^6)\,dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^7}{7} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{7} - 0 \right)$$

$$= \frac{4\pi}{21} \doteq 0.5984.$$

What about rotating about the y-axis?

First, isolate for x. $x = y^{\frac{1}{3}}$ and $x=y$

Rotate right most curve first.

$$\int_c^d \pi (x_R^2 - x_L^2)\,dy = \int_0^1 \pi \left((y^{\frac{1}{3}})^2 - y^2 \right) dy = \int_0^1 \pi \left(y^{\frac{2}{3}} - y^2 \right) dy$$

$$= \pi \left(\frac{3y^{5/3}}{5} - \frac{y^3}{3} \right) \Big|_0^1 = \pi \left(\frac{3}{5} - \frac{1}{3} - 0 \right) = \frac{4\pi}{15} \doteq 0.8378$$

these are the y-coord intersections.

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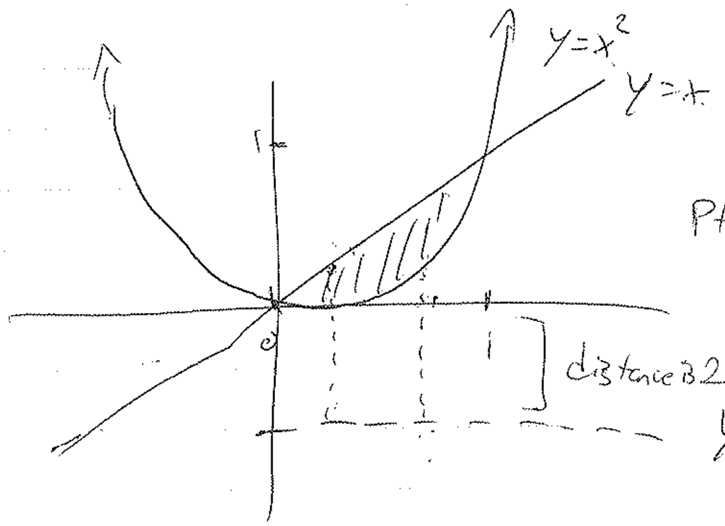
Ex: Find the volume of revolution of

$$y=x^2 \text{ and } y=x \quad \text{about } y=-2.$$

Sol'n:



Pts of intersection $(0,0)$ & $(1,1)$.

$$Vol = \int_0^1 \left(\pi(x+2)^2 - \pi(x^2+2)^2\right)dx$$

$$= \pi\int_0^1 (x^2+4x+4 - x^4-4x^2-4)\,dx$$

$$= \pi\int_0^1 (-x^4-3x^2+4x)\,dx$$

$$= \pi\left(\frac{-x^5}{5} - x^3 + 2x^2\right)\Big|_0^1$$

$$= \pi\left(-\frac{(1)^5}{5} - (1)^3 + 2(1)^2 - \left(-\frac{(0)^5}{5} - (0)^3 + 2(0)^2\right)\right)$$

$$= \pi\left(-\frac{1}{5} - 1 + 2\right)$$

$$= \frac{4\pi}{5}$$

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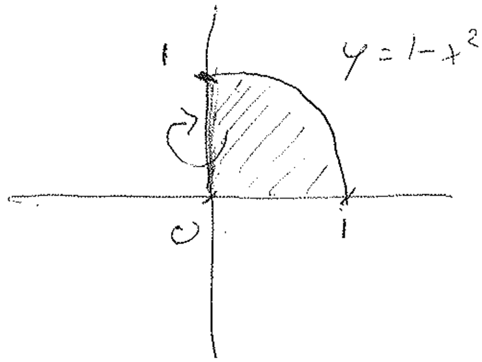
Volume of a paraboloid.

Q: A paraboloid is formed by rotating the curve $y = 1-x^2$ about the y-axis. Compute the volume of the paraboloid lying above the x-axis.



$x_L = 0$ $\quad$ x_R = right hand curve. Isolating for x in $y=1-x^2$ gives

$$x^2 = 1-y$$
$$x = \pm\sqrt{1-y}$$
$$x_R \longrightarrow \boxed{x=\sqrt{1-y}}$$

y-coord endpoints.

$$\text{Volume} = \int_0^1 \pi(x_R^2 - x_L^2)\,dy = \pi\int_0^1 \left((\sqrt{1-y})^2 - 0^2\right)dy$$

$$= \pi\int_0^1 (1-y)\,dy$$

$$= \pi\left(y - \frac{y^2}{2}\right)\Big|_0^1$$

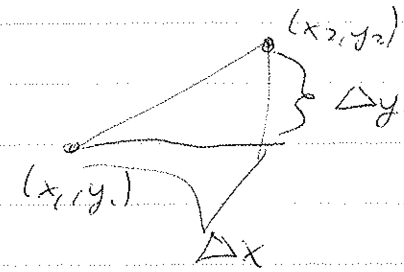
$$= \pi\left(1 - \frac{(1)^2}{2} - \left(0 - \frac{(0)^2}{2}\right)\right)$$

$$= \frac{\pi}{2}$$

Arc Length Jan 31/13

Compute the length of a piece of string.

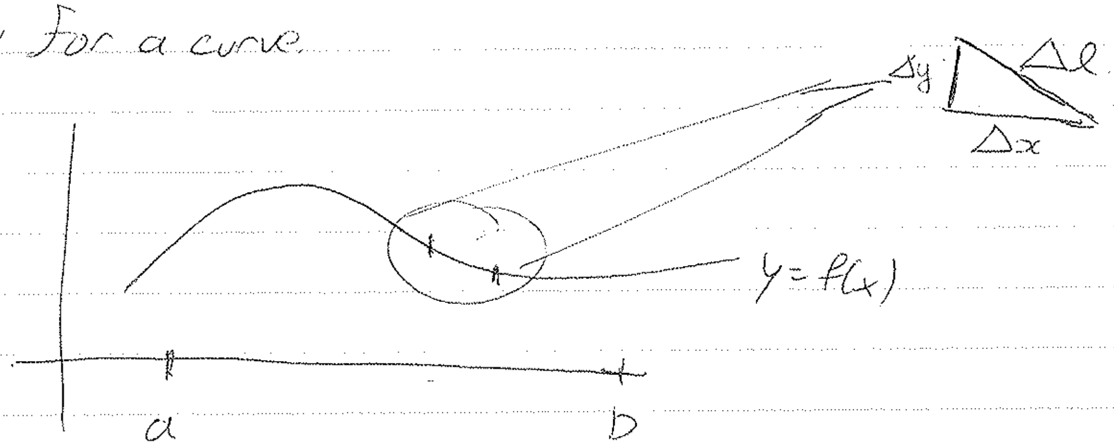
<u>Idea</u>: Recall. Distance between two points $P_1 = (x_1, y_1)$ & $P_2 = (x_2, y_2)$ is given by



$$
\begin{aligned}
D &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{\Delta x^2 + \Delta y^2} \\
&= |\Delta x| \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}
\end{aligned}
$$

Δx is a length hence positive $\rightsquigarrow \; = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$.

Now for a curve.



Break up the curve into little pieces then measure the lengths of those pieces. Locally,

$$\Delta l = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\; \Delta x$$

As the step size decreases, $\Delta x \to dx$ $\Delta y \to dy$ $\Delta l \to dl$.

$$dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (y')^2}\, dx = \sqrt{1 + (f'(x))^2}\, dx.$$

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Summing over all pieces gives

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\, dx.}$$ arc length.

Ex: Compute the length of the curve $y = \cos(x)$ between $x=0$ and $x = \frac{\pi}{2}$.

Sol'n. Note $y' = -\sin(x)$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{1+(-\sin(x))^2}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1+\sin^2(x)}\, dx$$

= Ask Wolfrom Alpha.

~~=1.90~~

$=1.9101\ldots$

Chapter 6: Differentials

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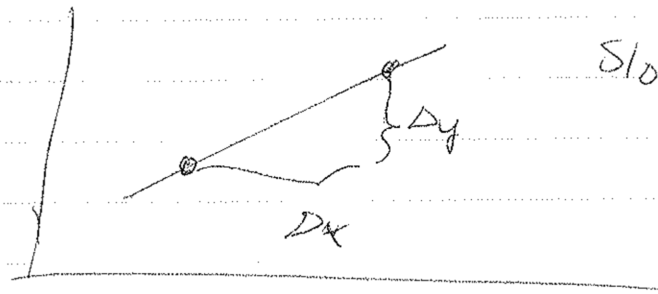
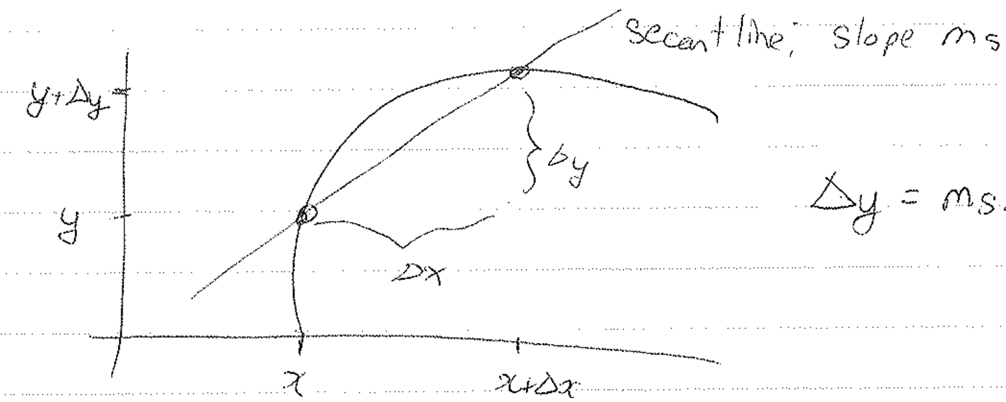
Recall:



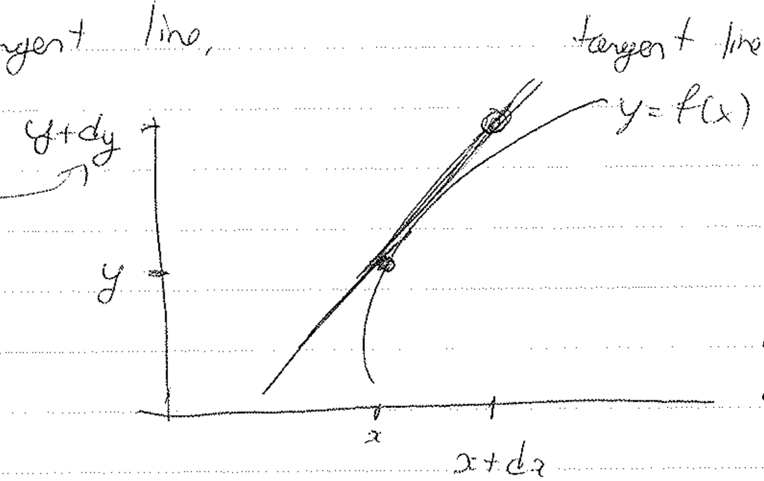
$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

On a curve.



$$\Delta y = m_s \Delta x$$

With a tangent line,



dy is a small value.

$$dy = m_t\, dx$$
or $df(x) = f'(x)\, dx$
or $df = f'(x)\, dx$
or $dy = f'(x)\, dx$

Recall $\dfrac{dy}{dx} = f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

The quantities dy & dx are called differentials.

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Two view points of derivatives:

(1) As a fraction of differentials $\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)dx$

(2) As an operator (ie a linear function on functions)

$$\frac{d}{dx}(f(x)) = f'(x).$$

Both are equally valid and get interchanged frequently.

Ex: If $y = f(x) = x^3$ then $f'(x) = 3x^2$ and $dy = 3x^2 dx$.

Ex: If $y = f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$ and $dy = \sec^2(x)\,dx$.

NB use $df = d(f(x)) = f'(x)\,dx$.

Rules of differentials:

~~Note~~ Let $C \in \mathbb{R}$ be a constant and $u(x), v(x)$ differentiable,

GO SLOW HERE! u & v look similar!

Operator view	Differential view	
1. $\frac{d}{dx}(C) = 0$	$dC = 0$	Linearity.
2. $\frac{d}{dx}(u(x) + v(x)) = \frac{du}{dx} + \frac{dv}{dx}$	$d(u+v) = du + dv$	
3. $\frac{d}{dx}(u(x)v(x)) = u\frac{dv}{dx} + v\frac{du}{dx}$	$d(uv) = u\,dv + v\,du$ (Flip)	(Product rule)
4. $\frac{d}{dx}(Cu(x)) = C\frac{du}{dx}$	$d(Cu) = C\,du$	

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Two view points of derivative.

(1) ~~Also~~ As a fraction of differentials $\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)dx$

(2) As a linear operator ie a linear function on functions.

$$\frac{d}{dx}(f(x)) = f'(x).$$

Both are equally valid and get interchanged frequently.

Ex: If $y = f(x) = x^3$ then $f'(x) = 3x^2$ and $dy = 3x^2dx$

Ex: If $y = TAN(x)$ then $f'(x) = SEC^2(x)$ and $dy = SEC^2(x)dx$.

NB: $df = d(f(x)) = f'(x)dx$

Rules for differentials: Let $c \in \mathbb{R}$ be a constant and $u(x), v(x)$ differentiable functions

Operator View	Differential view
(1) $\frac{d}{dx}(c) = 0$	$dc = 0$
(2) $\frac{d}{dx}(u(x) + v(x)) = \frac{du}{dx} + \frac{dv}{dx}$	$d(u+v) = du + dv$ (Linearity)
(3) $\frac{d}{dx}(cu(x)) = c\frac{du}{dx}$	$d(cu) = cdu$ (Linearity)
(4) $\frac{d}{dx}(u(x)v(x)) = u\frac{dv}{dx} + v\frac{du}{dx}$	$d(uv) = udv + vdu$ (Product Rule).

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Quiz

FORMULAS

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1) Evaluate $\displaystyle\int_0^2 |x-1|\,dx$

2) The growth rate of a plant is $g(t) = 36 - t^2$ for $0 \le t \le 6$ days

(a) Find the total growth over this period.

(b) Find the average growth rate over this period

(3) Compute $\displaystyle\frac{d}{dx}\int_5^x \ln(t)\,dt$ (where defined)

(4) Write in summation notation

$$3 + 7 + 11 + 15 + 19 + \cdots + 43$$

then evaluate

(5) Compute using Riemann Sums from the right

$$\int_1^5 x\,dx$$

No marks will be given for a solution using antiderivatives.

~~XXX~~ FORMULAS

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

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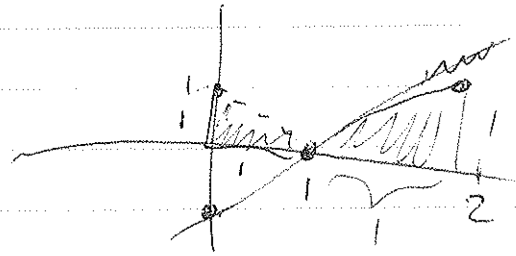
(1) $\int_0^2 |x-1|\,dx$



2pt

$$\int_0^2 |x-1|\,dx = \int_0^1 -(x-1)\,dx + \int_1^2 (x-1)\,dx \quad 1pt$$

$$= -\frac{x^2}{2} + x \Big|_0^1 + \left(\frac{x^2}{2} - x\right)\Big|_1^2 \quad 1pt$$

$$= \frac{-(1)^2}{2} + 1 - \left(\frac{-(0)^2}{2} + (0)\right) + \frac{(2)^2}{2} - 2 - \left(\frac{(1)^2}{2} - (1)\right) \quad 1pt$$

$$= \frac{-1}{2} + 1 - 0 + 2 - 2 - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1 \quad 2pt$$

OR $A = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1.$ 2pt.

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(2) Total growth $= \int_0^6 (36 - t^2)\,dt$ 1pt.

$= 36t - \frac{t^3}{3}\Big|_0^6$ 1 pt.

$= 36(6) - \frac{(6)^3}{3} - \left(36(0) - \frac{(0)^3}{3}\right)$

$= 216 - \frac{216}{3}$

$= 216 - 72$

$= 144.$ 1 pt.

(b) Average growth $= \frac{1}{6-0}\int_0^6 (36 - t^2)\,dt$ 1pt.

$= \frac{144}{6}$

$= 24.$ 1pt.

MAKE SURE x IS ON TOP!

FTC part 1.

(3) $\frac{d}{dx}\int_5^x \ln(t)\,dt = \boxed{\ln(x)}$ (where defined).

5pts

2 for $\ln(x) - \ln(5)$

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(4) $S = 3 + 7 + 11 + \cdots + 43$

Start at 3 then add 4 each time

$$S = \sum_{i=1}^{11} (3 + 4(i-1)) = \sum_{i=1}^{11} (3 + 4i - 4)$$ 2 for formula

$$= \sum_{i=1}^{11} (-1) + 4\sum_{i=1}^{11} i$$ 1 split

$$= -1(11-1+1) + 4\,\frac{(11)(11+1)}{2}$$ 1 for rules

$$= -11 + 2 \cdot 11 \cdot 12$$
$$= 11(-1 + 2 \cdot 12)$$
$$= 11(-1 + 24)$$
$$= 11 \cdot 23$$
$$= 253.$$ 1 formatting.

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(5) $\int_1^5 x\,dx$ $\qquad \Delta x = \frac{5-1}{n} = 4/n \quad = \frac{b-a}{n}$

$x_i = 1 + \frac{4i}{n} \quad = a + i\Delta x$ $\qquad$ 2 pts.

$$\int_1^5 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}+\frac{16}{n^2}i\right)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4}{n} + \sum_{i=1}^{n} \frac{16}{n^2}i\right) \qquad 1\text{ pt.}$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} 1 + \frac{16}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}(n) + \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right)\right) \qquad 1\text{ pt.}$$

$$= \lim_{n\to\infty} \left(4 + 8\,\frac{n}{n}\left(\frac{n+1}{n}\right)\right)$$

$$= \lim_{n\to\infty} \left(4 + 8\cdot 1\cdot\left(1+\underbrace{\tfrac{1}{n}}_{\to 0}\right)\right)$$

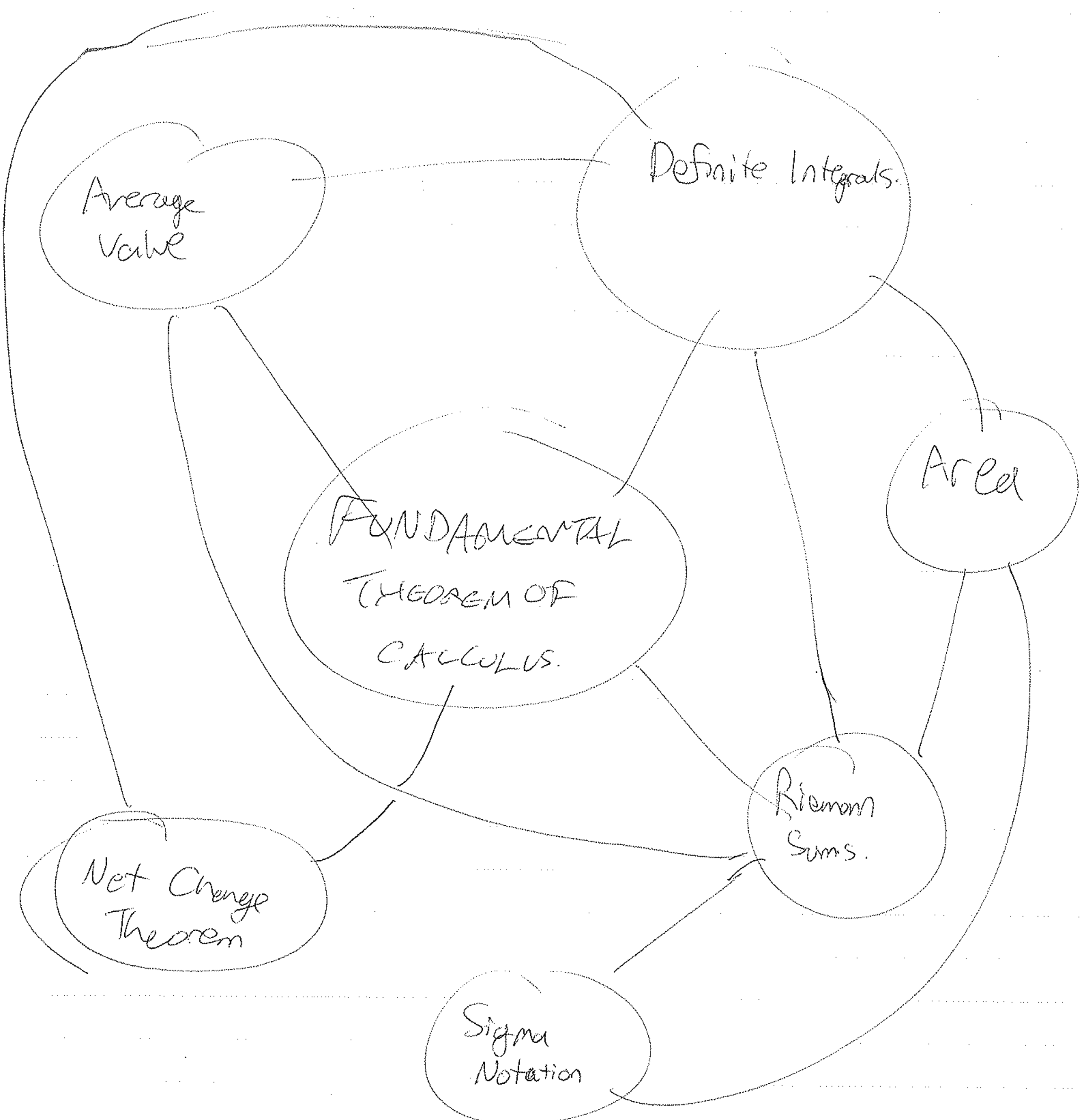
$$= 4+8$$

$$= 12 \qquad 1\text{ pt.}$$

Check: $\int_1^5 x\,dx = \frac{x^2}{2}\Big|_1^5 = \frac{(5)^2}{2} - \frac{(1)^2}{2} = \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = 12$ ✓

Jan 31/13

Do a Mind Map



$$\sum_{n=10}^{57} (-1)^n 2n$$

§6.2 Indefinite Integrals. Feb 5 / 13

Before: $\int_a^b f(x)\,dx$ = signed area = number (DEFINITE INTEGRAL)

Now, we will also use

$$\int f(x)\,dx = \text{antiderivative of } f(x) \quad \text{(INDEFINITE INTEGRAL)}$$

This will give us a family of functions.

Ex: $\int x^2\,dx = \frac{x^3}{3} + C$ $\qquad \int \frac{dx}{1+x^2} = \arctan(x) + C$

The linearity property of the definite integral still holds for indefinite integrals.

Ex: $\int (2x+1)\,dx = 2\int x\,dx + \int (1)\,dx = 2\left(\frac{x^2}{2}\right) + x + C = x^2+x+C$

Substitution Rule For Indefinite Integrals.

Idea: Compute $\int (x+1)^{10}\,dx$.

As of right now, we have no choice but to either guess the antiderivative OR expand this out and integrate. Notice that $(x+1)^{10}$ is a composition of $g(x)=x+1$ and $f(x)=x^{10}$ so $f(g(x))=(x+1)^{10}$.

We want to "undo" the chain rule.

Feb. 5/13

Abstraction

Let $F(x)$ & $g(x)$ be differentiable functions. Let $u = g(x)$.

Chain Rule: $\frac{d}{dx} F(g(x)) = F'(g(x))\,g'(x)$

Integrating gives

FTC still works

$$\int F'(g(x))\,g'(x)\,dx = \int \frac{d}{dx} F(g(x))\,dx \overset{\downarrow}{=} F(g(x)) + C = F(u) + C$$

$$= \int \frac{d}{du} F(u)\,du$$

$$= \int F'(u)\,du$$

Theorem: (Substitution Rule for Indefinite Integrals).

If $u = g(x)$ is differentiable with range I (I is an interval) and $f(x)$ is a continuous function on I. Then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Ex: $\int (x+1)^{10}\,dx$ Let $u = x+1 \Rightarrow du = dx$. Then

$$\int (x+1)^{10}\,dx = \int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(x+1)^{11}}{11} + C$$

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Ex: $\int \frac{x}{1+x^2}\,dx$ Let $u = 1+x^2$ so $du = 2x\,dx$ $\frac{du}{2} = x\,dx$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2| + C$$

Ex: $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx$

Try $u = \cos(x)$ $du = -\sin(x)\,dx$.

$$\int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{u}{\sin(x)}\frac{du}{(-\sin(x))}$$ PROBLEM we have x values & u values.

Conclusion: Made a bad substitution. Use $u = \sin(x) \Rightarrow du = \cos(x)dx$

$$\int \frac{\cos(x)\,dx}{\sin(x)} = \int \frac{du}{u} = \ln|u| + C = \ln|\sin(x)| + C.$$

What about definite integrals?

Theorem: (Substitution Rule For Definite Integrals).

Let $u = g(x)$ be a differentiable function with continuous derivative, range an interval I and let $f(u)$ be a function that is continuous on I. Then for $a,b \in \mathbb{R}$, we have

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

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Ex: $\int_0^{\pi} \sin(x) e^{\cos(x)} dx$ Let $u = \cos(x)$ $du = -\sin(x)\,dx$.

$u(0) = \cos(0) = 1$ $u(\pi) = \cos(\pi) = -1$

$$\int_{x=0}^{\pi} \sin(x) e^{\cos(x)} dx = \int_{u=1}^{-1} e^u(-du) = -\int_1^{-1} e^u du = \int_{-1}^{1} e^u du.$$

$$= e^u \Big|_{-1}^{1} = e^1 - e^{-1}$$

$\boxed{\text{WARNING}}$

Substitution rule for indefinite integrals.

After plugging in u & integrating, you MUST plug in the original variable

Substitution rule for definite integrals.

After plugging in u, you MUST change the endpoints to match the variable u. After this you DO NOT plug back in your original variable.

A wrong example:

Let $u = 1+x^3$ so $du = 3x^2 dx$

BAD no need for original variable if you change endpoints

$$\int_0^1 \frac{3x^2}{1+x^3} dx = \int_0^1 \frac{du}{u} = \ln|u| \Big|_0^1 = \ln|1+x^3| \Big|_0^1 = \ln(2) - \ln(1) = \ln(2)$$

BAD should change endpoints!

$u(0) = 1$ & $u(1) = 2$

BAD don't plug in original.

$$\int_0^1 \frac{3x^2}{1+x^3} dx = \int_1^2 \frac{du}{u} = \ln|u| \Big|_1^2 = \ln|1+x^3| \Big|_1^2 = \ln(9) - \ln(2).$$

CORRECT WAY $\int_0^1 \frac{3x^2}{1+x^3} dx = \int_1^2 \frac{du}{u} = \ln|u| \Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$

$\checkmark\checkmark$

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Cases where the substitution is unclear. Sometimes you need tricks.

Trick 1: Multiply by 1. (Very hard trick).

Ex: $\int \sec(x)\,dx$.

Trick: Multiply top & bottom by $\sec(x)+\tan(x)$.

$$\int \sec(x)\,dx = \int \sec(x)\,\frac{(\sec(x)+\tan(x))}{(\sec(x)+\tan(x))}\,dx = \int \left(\frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\right)dx$$

Let $u=\sec(x)+\tan(x)$ then $du = \sec(x)\tan(x)+\sec^2(x)$

$$\int \sec(x)\,dx = \int \frac{du}{u} = \ln|u|+C = \ln|\sec(x)+\tan(x)|+C.$$

Trick 2: Perfect Square in Denominator.

Ex: $\displaystyle\int \frac{dx}{x^2-6x+9} = \int \frac{dx}{(x-3)^2}$ Let $u=x-3$ $du=dx$

$$= \int \frac{du}{u^2} = -u^{-1}+C = -(x-3)^{-1}+C.$$

Trick 3 : Complete the square.

$$\int \frac{dx}{x^2-6x+18} = \int \frac{dx}{1(x^2-6x+\overbrace{9}^{(\frac{1}{2}b)^2}-9)+18} = \int \frac{dx}{(x-3)^2-9+18} = \int \frac{dx}{(x-3)^2+9}$$

$$= \frac{1}{9}\int \frac{dx}{\frac{(x-3)^2}{9}+1}$$

$$= \frac{1}{9}\int \frac{dx}{\left(\frac{(x-3)}{3}\right)^2+1} \qquad \text{Let } u = \frac{x-3}{3} \qquad du = \frac{dx}{3}$$

$$= \frac{1}{9}\int \frac{3\,du}{u^2+1}$$

$$= \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan\left(\frac{x-3}{3}\right) + C$$

Note: Look at $\int \frac{dx}{x-x^2}$ Changing one sign can DRASTICALLY change the way you should solve an integral.

$$\int \frac{dx}{(x-x^2)} = \int\left(\frac{-1}{x} + \frac{1}{x-1}\right)dx$$

$$= -\ln|x| + \ln|x-1| + C.$$

Trig Identities

Feb. 5/13.

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$
3. $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
4. $\sin(2x) = 2\sin(x)\cos(x)$ let $A=B=x$.
5. $\cos(2x) = \cos^2(x) - \sin^2(x)$ let $A=B=x$.

* 6. $\cos^2(x) = (1+\cos(2x))/2$

* 7. $\sin^2(x) = (1-\cos(2x))/2$

8. $\tan^2(x) + 1 = \sec^2(x)$ Divide (1.) by $\cos^2(x)$.

To derive 6 & 7 from 1 to 5

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{(By 5.)}$$
$$\cos(2x) = \cos^2(x) - (1-\cos^2(x)) \quad \text{(By 1.)}$$
$$\cos(2x) = \cos^2(x) - 1 + \cos^2(x)$$
$$\cos(2x) + 1 = 2\cos^2(x)$$
$$\frac{\cos(2x)+1}{2} = \cos^2(x) \quad \leftarrow \text{eqn 6.}$$
$$\frac{\cos(2x)}{2} + \frac{1}{2} = 1 - \sin^2(x) \quad \text{(By 1.)}$$
$$\sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2} \quad \leftarrow \text{eqn 7.}$$

<u>NB</u> 4-7 are called double angle formulas

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Practice

1. $\int \frac{3}{x+4}\,dx$

2. $\int_0^1 x^2 e^{x^3}\,dx$

3. $\int \frac{dx}{(x+1)^2+1}$

(4) $\int (x+2)\sqrt{x^2+4x-1}\,dx$

5. $\int_0^{\pi} \cos^5(x)\sin(x)\,dx$

6. $\int \frac{3x^2}{1+x^6}\,dx$

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Let $u = x+4 \quad du = dx$

1. $\displaystyle\int \frac{3}{x+4}\,dx = \int \frac{3}{u}\,du = 3\ln|u| + C = 3\ln|x+4| + C.$

Let $u = x^3$ so $du = 3x^2dx \quad u(0)=0 \quad u(1)=1$

2. $\displaystyle\int_0^1 x^2 e^{x^3}\,dx \overset{\checkmark}{=} \int_0^1 e^u\left(\frac{du}{3}\right) = \frac{1}{3}e^u\Big|_0^1 = \frac{1}{3}(e^1 - e^0) = \frac{1}{3}(e-1).$

Let $u = x+1 \quad du = dx$

3. $\displaystyle\int \frac{dx}{(x+1)^2+1} \overset{\checkmark}{=} \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+1) + C.$

$u = x^2+4x-1 \quad du = (2x+4)dx = 2(x+2)dx$

4. $\displaystyle\int (x+2)\sqrt{x^2+4x-1}\,dx \overset{\checkmark}{=} \int \sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + C$

$\displaystyle = \frac{(x^2+4x-1)^{3/2}}{3} + C.$

let $u = \cos(x) \quad du = -\sin(x)dx \quad u(0) = 1 \quad u(\pi) = -1$

5. $\displaystyle\int_0^{\pi} \cos^5(x)\sin(x)\,dx \overset{\checkmark}{=} \int_1^{-1} u^5(-du) = \int_{-1}^1 u^5\,du = \frac{u^6}{6}\Big|_{-1}^1$

$\displaystyle = \frac{1^6}{6} - \frac{(-1)^6}{6} = 0.$

~~let $u = 1+x^6 \quad du = 6x^5dx = (3x^2)(2x^3)dx$~~

6. $\displaystyle\int \frac{3x^2}{1+x^6}\,dx = \cancel{\int \frac{3x^2}{1+(x^3)^2}dx}$

$\displaystyle = \cancel{\int \frac{1}{u}\left(\frac{du}{2x^3}\right)}$ ⇐ We have ~~problems.~~

Let $u = x^3. \quad du = 3x^2dx$

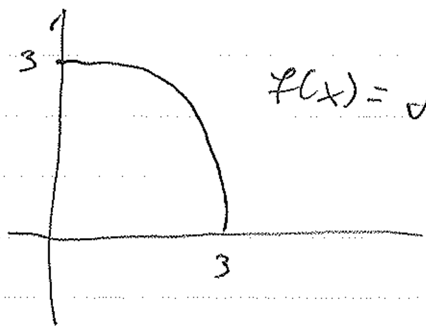
$\displaystyle = \int \frac{3x^2}{1+(x^3)^2}\,dx \overset{\checkmark}{=} \int \frac{du}{1+u^2} = \arctan(u) + C$

$= \arctan(x^3) + C.$

Application: Two Dimensional Centre of Mass WITH Uniform Density ρ

Feb 7/13

Ex:



$$f(x) = \sqrt{9-x^2} = y$$

$$9 - x^2 = y^2$$
$$9 - y^2 = x^2$$
$$\sqrt{9-y^2} = x \doteq g(y)$$

WANT POSITIVE

$$M = \rho A$$

Recall: $\bar{x} = \frac{1}{M}\int_a^b \rho x f(x)\,dx = \frac{\rho}{\rho A}\int_a^b x f(x)\,dx = \frac{1}{A}\int_a^b x f(x)\,dx.$

$$\bar{y} = \frac{1}{M}\int_c^d \rho y g(y)\,dy = \frac{\rho}{\rho A}\int_c^d y g(y)\,dy = \frac{1}{A}\int_c^d y g(y)\,dy$$

Alternate definition for $\bar{y}$.

$$\bar{y} = \frac{1}{A}\int_a^b \frac{f(x)^2}{2}\,dx$$

Think of this as the mass $\rho f(x)$ times half the distance from the x-axis $\frac{f(x)}{2}$. Integrate then normalize by mass ρA.

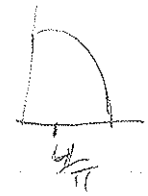
We call $(\bar{x}, \bar{y})$ the ~~centroid~~ centroid

For this example,

$$A = \int_0^3 \sqrt{9-x^2}\,dx = \frac{\pi(3)^2}{4} = \frac{9}{4}\pi.$$

$$\bar{x} = \frac{1}{A}\int_0^3 x\sqrt{9-x^2}\,dx \qquad u = 9-x^2 \quad du = -2x\,dx$$
$$u(0) = 9 \qquad u(3) = 0$$

$$= \frac{1}{\frac{9\pi}{4}}\int_9^0 \sqrt{u}\left(\frac{du}{-2}\right) = \frac{2}{9\pi}\int_9^0 -\sqrt{u}\,du = \frac{2}{9\pi}\int_0^9 \sqrt{u}\,du$$



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$$\bar{x} = \frac{2}{9\pi}\left(\frac{2u^{3/2}}{3}\right)\Big|_0^9 = \frac{4}{27\pi}\left(9^{3/2} - 0\right) = \frac{4}{\pi}.$$

$$\bar{y} = \frac{1}{A}\int_0^3 y\sqrt{9-y^2}\,dy = \bar{x} = \frac{4}{\pi}$$

Exercise:

$$\bar{y} = \frac{1}{A}\int_0^3 \frac{\left(\sqrt{9-x^2}\right)^2}{2}\,dx = \cdots = \frac{4}{\pi}$$

Trig Integrals. Feb 7/13

Goal: Integrate $\int \sin^a(x)\cos^b(x)\,dx$ & $\int \tan^a(x)\sec^b(x)\,dx$.

Ex: $\int \sin^5(x)\,dx$ Odd power of $\sin(x)$ [OR of $\cos(x)$]

Idea: Split off a $\sin(x)$, and use $\sin^2(x) = 1 - \cos^2(x)$.

$$\int \sin^5(x)\,dx = \int \sin^4(x)\sin(x)\,dx = \int (\sin^2(x))^2 \sin(x)\,dx$$

$$= \int (1-\cos^2(x))^2 \sin(x)\,dx$$

Let $u = \cos(x)$

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$$= \int (1-u^2)^2(-du)$$

$$= -\int (1 - 2u^2 + u^4)\,du$$

$$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + C$$

$$= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{\cos^5(x)}{5} + C$$

Ex: $\int \cos^4(x)\,dx$ Even power of $\sin(x)$ OR $\cos(x)$.

IDEA: Use double angle formulas.

$$\int \cos^4(x)\,dx = \int (\cos^2(x))^2\,dx$$

$=$

$$\boxed{\cos^2(x) = \cancel{\cos^2(x) - \sin^2(x)} = \frac{1+\cos(2x)}{2}}$$

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$$\int \cos^4(x)\,dx = \int (\cos^2(x))^2\,dx \qquad \text{Use } \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \int \left(\frac{1}{4} + \frac{\cos(2x)}{2} + \frac{\cos^2(2x)}{4}\right) dx$$

ANOTHER $\cos^2(x)$.

$$= \int \left(\frac{1}{4} + \frac{\cos(2x)}{2} + \frac{1+\cos(4x)}{8}\right) dx \qquad \cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \int \left(\frac{3}{8} + \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8}\right) dx$$

$$= \frac{3}{8}x + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C.$$

Strategy for $\int \sin^m(x)\cos^n(x)\,dx$ Feb 7/13

(a) If the power of $\sin(x)$ (or $\cos(x)$) is odd, save a power of $\cos(x)$ (OR $\sin(x)$) and change the other powers using $\sin^2(x) = 1-\cos^2(x)$ (OR $\cos^2(x) = 1-\sin^2(x)$). Then substitute $u=\cos(x)$ (OR $u=\sin(x)$).

Ex: If m is an odd integer.

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{m-1}(x)\cos^n(x)\sin(x)\,dx$$
$$= \int (1-\cos^2(x))^{\frac{m-1}{2}}\cos^n(x)\sin(x)\,dx$$ Then let $u=\cos(x)$.

(Similar if n is odd).

(b) If both powers are even then use double angle formulas.

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$$
$$\sin(x)\cos(x) = \tfrac{1}{2}\sin(2x).$$

KEY IDEA: $\sin(x)$ & $\cos(x)$ are derivatives of each other.

Next Step: Try with $\tan(x)$ & $\sec(x)$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

$\int \tan^a(x) \sec^b(x)\,dx$. Feb. 12/13

$\tan^2(x)+1 = \sec^2(x)$.

Recall: $\frac{d}{dx}\tan(x) = \sec^2(x)$ & $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$.

Ex: $\int \tan^2(x)\sec^4(x)\,dx = \int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$

$= \int \tan^2(x)(1+\tan^2(x))\sec^2(x)\,dx$

Let $u = \tan(x)$, $du = \sec^2(x)\,dx$

$= \int u^2(1+u^2)\,du$

$= \int (u^2+u^4)\,du$

$= \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C$.

Ex: $\int \tan^3(x)\sec^7(x)\,dx = \int \tan^2(x)\sec^6(x)\,(\sec(x)\tan(x))\,dx$.

$= \int (\sec^2(x)-1)\sec^6(x)(\sec(x)\tan(x))\,dx$.

Let $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$

$= \int (u^2-1)u^6\,du$

$= \int (u^8-u^6)\,du$

$= \frac{u^9}{9} - \frac{u^7}{7} + C$

$= \frac{(\sec(x))^9}{9} - \frac{(\sec(x))^7}{7} + C$

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Strategy for $\int \tan^m(x)\sec^n(x)\,dx$ m, n are integers. (positive)

(a) If the power of $\sec(x)$ is even, so $\boxed{n = 2K}$, save a $\sec^2(x)$ and use ~~the~~ $\sec^2(x) = 1 + \tan^2(x)$ and $u = \tan(x)$.

$$\int \tan^m(x)\sec^{2K}(x)\,dx = \int \tan^m(x)\left(\sec^2(x)\right)^{K-1}\sec^2(x)\,dx$$
$$= \int \tan^m(x)\left(1 + \tan^2(x)\right)^{K-1}\sec^2(x)\,dx.$$

Then let $u = \tan(x)$. $= \ldots$

(b) If $\tan(x)$ is raised to an odd power, so $m = 2K+1$, save a $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$. & let $u = \sec(x)$.

$$\int \tan^{2K+1}(x)\sec^n(x)\,dx = \int \left(\tan^2(x)\right)^K \sec^{n-1}(x)\left(\sec(x)\tan(x)\right)dx.$$
$$= \int \left(\sec^2(x) - 1\right)^K \sec^{n-1}(x)\left(\sec(x)\tan(x)\right)dx$$

Let $u = \sec(x)$. $= \ldots$

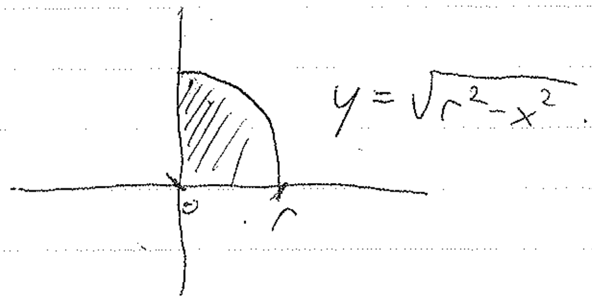
__NB__ For the other cases, no general algorithm.

Ex: $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$ (mult by 1 trick).

let $u = \cos(x)$ $du = -\sin(x)\,dx$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$
$$= \ln|\cos(x)^{-1}| + C$$
$$= \ln|\sec(x)| + C.$$

[Inverse] Trig substitutions. Feb. 12/13



Area of circle $= 4\int_0^r \sqrt{r^2-x^2}\,dx = \pi r^2$. ← why?

IDEA: $\sqrt{r^2 - r^2\sin^2\theta} = |r|\sqrt{1-\sin^2\theta} = |r|\sqrt{\cos^2\theta} = |r||\cos\theta|$. (✡)

So we substitute $x = r\sin\theta$ $\quad dx = r\cos\theta\, d\theta$.

Endpoints. $x=0$, $\quad 0 = r\sin\theta \Rightarrow 0=\sin\theta \Rightarrow \theta = 0$.
$\qquad x = r$, $\quad r = r\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}$

Thus, $4\int_0^r \sqrt{r^2-x^2}\,dx = 4\int_0^{\pi/2} \sqrt{r^2-(r\sin\theta)^2}\,(r\cos\theta\, d\theta)$

Use (✡) $\Rightarrow = 4\int_0^{\pi/2} |r\cos\theta|\,(r\cos\theta\, d\theta)$

$$= 4\int_0^{\pi/2} r^2\cos^2\theta\, d\theta$$

$$= 4r^2\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= 2r^2\int_0^{\pi/2} (1+\cos(2\theta))\,d\theta$$

$$= 2r^2\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/2}$$

$$= 2r^2\left(\frac{\pi}{2} + \cancel{\frac{\sin(\pi)}{2}}^{0} - 0 - 0\right) = \pi r^2$$

Table of Substitutions Feb. 12/13

Expression	Substitution (Intervals where function is one to one ie has an inverse)		Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Ex: $\displaystyle\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{x^2+4x+4-4+7} = \int \frac{dx}{(x+2)^2+3}$

Let $x+2 = \sqrt{3}\tan\theta$ so $dx = \sqrt{3}\sec^2\theta\, d\theta$

$$\int \frac{dx}{x^2+4x+7} = \int \frac{\sqrt{3}\sec^2\theta\, d\theta}{(\sqrt{3}\tan\theta)^2+3} = \frac{\sqrt{3}}{3}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1}$$

$$= \frac{\sqrt{3}}{3}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \frac{\sqrt{3}}{3}\int d\theta = \frac{\sqrt{3}}{3}\theta + C$$

Since $x+2 = \sqrt{3}\tan\theta$, we have $\frac{x+2}{\sqrt{3}} = \tan\theta$

hence $\theta = \arctan\left(\frac{x+2}{\sqrt{3}}\right)$

Thus, $\displaystyle\int \frac{dx}{x^2+4x+7} = \frac{\sqrt{3}}{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C.$

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Ex: $\int \frac{dx}{x^2\sqrt{x^2-16}}$ Let $x = 4\sec\theta$

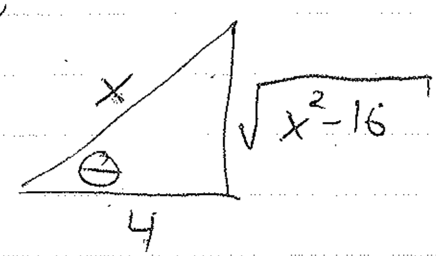
$dx = 4\sec\theta \tan\theta \, d\theta.$

$$= \int \frac{4\sec\theta \tan\theta \, d\theta}{16\sec^2\theta \sqrt{16\sec^2\theta - 16}} = \int \frac{\tan\theta \, d\theta}{4\sec\theta\,(4\sqrt{\sec^2\theta - 1})}$$

$$= \int \frac{\tan\theta \, d\theta}{16\sec\theta\sqrt{\tan^2\theta}} = \int \frac{d\theta}{16\sec\theta} = \frac{1}{16}\int \cos\theta \, d\theta = \frac{1}{16}\sin\theta + C.$$

Need to change $\sin\theta$ to x. We know

$$\frac{x}{4} = \sec\theta = \frac{H}{A}$$



Thus, $\sin\theta = \frac{O}{H} = \frac{\sqrt{x^2-16}}{x}.$

$$\int \frac{dx}{x^2\sqrt{x^2-16}} = \frac{1}{16}\sin\theta + C = \frac{1}{16}\,\frac{\sqrt{x^2-16}}{x} + C$$

Ex: $\int \frac{2x}{\sqrt{x^2+1}}\,dx$ Let $u = x^2+1$

$du = 2x\,dx$

$$= \int \frac{du}{\sqrt{u}} = 2u^{\frac{1}{2}} + C = 2\sqrt{x^2+1} + C.$$

Moral: Sometimes other methods are easier to solve integrals than others!

Integration By Parts.

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Let's undo the product rule.

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$$

$$duv = \cancel{}\, udv + vdu$$

~~Em~~
Integrate bothsides

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx.$$

$$uv = \int udv + \int vdu$$

Rearranging gives

Theorem (Integration by Parts).

Let $f(x)=u$ and $g(x)=v$ be differentiable functions. Then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

$$\int udv = uv - \int vdu$$

Ex: $\int xe^x dx$. Q: which function do I make my u & my v?

"table method"

$$\begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x dx \end{array}$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

INTEGRATION BY PARTS.

What if we did $\quad u = e^x \quad v = \frac{x^2}{2}$?

$\quad du = e^x dx \quad dv = x\,dx$

$$\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x dx$$ STILL hard to integrate!

Ex: $\int_0^{\pi} x\cos(x)\,dx \qquad u = x \quad v = \sin(x)$

$\qquad du = dx \quad dv = \cos(x)dx$

$$\int_0^{\pi} x\cos(x)\,dx = x\sin(x)\Big|_0^{\pi} - \int_0^{\pi} \sin(x)\,dx$$

$$= \cancel{\pi\sin(\pi) - 0\sin(0)} - (-\cos(x))\Big|_0^{\pi}$$

$$= \cos(\pi) - \cos(0)$$

$$= (-1) - (1)$$

$$= -2$$

Ex: $\int \ln(x)\,dx \qquad u = \ln(x) \quad v = x$

$\qquad du = \frac{1}{x}dx \quad dv = dx$

$$\int \ln(x)\,dx = x\ln(x) - \int \cancel{x}\left(\frac{1}{\cancel{x}}\right)dx = x\ln(x) - \int dx = x\ln(x) - x + C$$

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Last time: Recall integration by parts.

$$\int u\,dv = uv - \int v\,du$$

Rule of thumb. Make the u function

u L Logarithms ~~xxxx~~ $\ln x$

I Inverse trig function $\arctan x$, $\arcsin x$

A Algebraic Function x^3, $3x$

T Trig Functions $\sin(x)$, $\cos(x)$

E Exponentials e^x, 5^x.

D dv

Ex: $\int x^6 \ln(x)\,dx$ Try int by parts $u = \ln(x)$ $v = \frac{x^7}{7}$

$du = \frac{dx}{x}$ $dv = x^6 dx$

$$\int x^6 \ln(x)\,dx = \frac{x^7}{7}\ln(x) - \int \frac{x^7}{7}\left(\frac{dx}{x}\right) = \frac{x^7}{7}\ln(x) - \int \frac{x^6}{7}\,dx$$

$$= \frac{x^7}{7}\ln(x) - \frac{1}{7}\left(\frac{x^7}{7}\right) + C$$

$$= \frac{x^7}{7}\ln(x) - \frac{x^7}{49} + C$$

Ex: $\int_0^1 \arctan(x)\,dx$ Int. by parts $u = \arctan(x)$ $v = x$

$du = \frac{1}{x^2+1}dx$ $dv = dx$

$$\int_0^1 \arctan(x)\,dx = x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$

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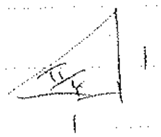
$$\int_0^1 \arctan(x)\,dx = \arctan(1) - \int_1^2 \frac{1}{w}\left(\frac{dw}{2}\right)$$

Let $w = x^2+1$ $\quad w(0)=1$

$dw = 2x\,dx$ $\quad w(1)=2$

$$= \arctan(1) - \frac{1}{2}\left(\ln|w|\right)\Big|_1^2$$



$$= \arctan(1) - \left(\frac{1}{2}\ln(2) - \frac{1}{2}\ln(1)\right)$$

$\tan\left(\frac{\pi}{4}\right)=1$

$\arctan(1)=\frac{\pi}{4}$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

Ex: $\displaystyle\int \frac{\ln(x)}{x}\,dx$ DON'T DO PARTS! Let $u=\ln(x)$

$du = \frac{1}{x}\,dx$

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\ln(x)^2}{2}+C.$$

Ex: $\displaystyle\int t^2 e^t\,dt.$ Int by Parts $u=t^2 \longrightarrow v=e^t$

$du = 2t\,dt \longleftarrow dv = e^t\,dt$

$$\int t^2e^t\,dt = t^2e^t - \int 2te^t\,dt$$ ← still hard. Use Parts again!

$w = t \longrightarrow x = e^t$

$dw = dt \longleftarrow dx = e^t\,dt$

$$= t^2e^t - 2\left(te^t - \int e^t\,dt\right)$$

$$= t^2e^t - 2te^t + 2e^t + C. \checkmark$$

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<u>Ex</u>: $\int \sin(t)\, e^t \, dt$

$u = \sin(t) \quad v = e^t$
$du = \cos(t) \quad dv = e^t dt$

$$\int \sin(t) e^t dt = \sin(t) e^t - \int \cos(t) e^t dt$$

$w = \cos(t) \quad x = e^t$
$dw = -\sin(t) \quad dx = e^t dt$

$$= \sin(t) e^t - \left(\cos(t) e^t - \int e^t(-\sin(t))\, dt\right)$$

$$= \sin(t) e^t - \cos(t) e^t - \int \sin(t) e^t dt$$

$$2\int \sin(t) e^t dt = \sin(t) e^t - \cos(t) e^t + C$$

$$\int \sin(t) e^t dt = \frac{\sin(t) e^t - \cos(t) e^t}{2} + D \qquad \text{where } D = \frac{C}{2}$$

Integration By Partial Fractions. Feb. 14/13

Recall $\displaystyle\int \frac{dx}{x+a} = \ln|x+a| + C$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{(\frac{x}{a})^2+1} = \frac{1}{a^2}\left(a \arctan\left(\frac{x}{a}\right)\right) + C = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

<u>Ex</u>: $\displaystyle\int \frac{(3x+1)}{x^2+2x-3}\,dx$ ⇐ Seems hard

$$\int\left(\frac{1}{x-1} + \frac{2}{x+3}\right)dx = \ln(x-1) + 2\ln(x+3) + C$$ ⇐ easy

NB $\displaystyle\frac{1}{x-1} + \frac{2}{x+3} = \frac{(1)(x+3) + 2(x-1)}{(x-1)(x+3)} = \frac{x+3+2x-2}{x^2-2x-3} = \frac{3x+1}{x^2+2x-3}$

<u>Question</u> How do you go from an "ugly" rational polynomial function $f(x) = \dfrac{P(x)}{Q(x)}$

to a "nice" rational function.

(1) If degree of $P(x) \geq$ degree of $Q(x)$, do long division (OMITTED)

(2) Simplify into smaller pieces depending on the factorization of the denominator.

Ex: If $Q(x) = (x-3)^3(x+2)(x^2+x+4)^2$, then write

↖ irreducible

$$\frac{P(x)}{Q(x)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x+2)} + \frac{Ex+F}{(x^2+x+4)} + \frac{Gx+H}{(x^2+x+4)^2}$$

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(3) Find $A, \dots H$.
(4) Integrate term by term.

NB We will only worry about $Q(x)$ = quadratic polynomial.
The above is for reference only.

Let's try this with our example.

$$\int \frac{3x+1}{x^2+2x-3}\,dx \qquad \text{Step (1)} \checkmark$$

Step (2) $\dfrac{3x+1}{x^2+2x-3} = \dfrac{3x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3} = \dfrac{A(x+3)+B(x-1)}{(x-1)(x+3)}$

Method 1: Plug in x-values.

Since denominators are the same, the numerators must be the same. So

$$3x+1 = A(x+3) + B(x-1)$$

This holds for ALL x values. For example, if $x=-3$, then

$$-8 = 3(-3)+1 = A((-3)+3) + B((-3)-1) = -4B \quad \text{Thus, } B=2.$$

Also, when $x=1$, we have

$$4 = 3(1)+1 = A((1)+3) + B((1)-1) = 4A \quad \text{Thus, } A=1.$$

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$$\frac{3x+1}{x^2+2x-3} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

Method 2: Comparing coefficients.

As above, $3x+1 = A(x+3)+B(x-1) = Ax+3A+Bx-B = (A+B)x+3A-B$

Thus $3 = A+B$ [1] & $1 = 3A-B$ [2]

[1]+[2]: $4 = 4A$ so $A=1$

$A=1$ in [1]: $3 = 1+B$ so $B=2$.

$$\int \frac{3x+1}{x^2+2x-3}\,dx = \int\left(\frac{A}{x-1}+\frac{B}{x+3}\right)dx = \int\left(\frac{1}{x-1}+\frac{2}{x+3}\right)dx = \ln|x-1| + 2\ln|x+3| + C$$

Ex: $$\int \frac{3x+5}{x^2+2x-3}\,dx = \int \frac{3x+5}{(x-1)(x+3)}\,dx = \int\left(\frac{A}{x-1}+\frac{B}{x+3}\right)dx$$
$$= \int \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

So $3x+5 = A(x+3)+B(x-1)$

when $x=1$, we get $8=4A$ so $A=2$
when $x=-3$ we get $-4=-4B$ so $B=1$.

$$\int \frac{3x+5}{x^2+2x-3}\,dx = \int\left(\frac{2}{x-1}+\frac{1}{x+3}\right)dx = 2\ln|x-1| + \ln|x+3| + C$$

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Ex: $\int \frac{x+1}{x^2-2x+5}\,dx$

Key complete the square. $x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4$.

$= \int \frac{x+1}{(x-1)^2+4}\,dx$ Let $u = x-1$ $x = u+1$
$du = dx$

$= \int \frac{(u+1)+1}{u^2+4}\,du$

$= \int \frac{u+2}{u^2+4}\,du$

$= \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du$ Let $w = u^2+4$ in the first integral.
$dw = 2u\,du$

$= \int \frac{1}{w}\frac{dw}{2} + 2\int \frac{1}{u^2+4}\,du$

$= \frac{1}{2}\ln|w| + 2\left(\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right) + C$

$= \frac{1}{2}\ln|w| + \arctan\left(\frac{u}{2}\right) + C$

$= \frac{1}{2}\ln|u^2+4| + \arctan\left(\frac{u}{2}\right) + C$

$= \frac{1}{2}\ln|(x-1)^2+4| + \arctan\left(\frac{x-1}{2}\right) + C.$

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Strategy For Integration

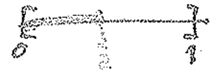
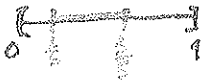
0. Just do it Eg $\int \frac{dx}{x}$
1. Simplify integrand Eg. $\int \sqrt{x}(1+\sqrt{x})\,dx$
2. Try a substitution Eg $\int \frac{x\,dx}{x^2+1}$
3. Classify the integrand
   (a) Trig functions Eg. $\int TAN^3(x)\,SEC^7(x)\,dx$
   (b) Rational functions Eg. $\int \frac{x^2+x+1}{(x-1)^3}\,dx$ Use partial fraction. Also complete square.
   (c) Integration By Parts Eg $\int x\ln x\,dx$
   (d) Radicals. Eg $\int \frac{dx}{x^2\sqrt{x^2-9}}$ Use trig sub.
4. Repeat the above (try non-obvious method)

   Eg. $\int \frac{dx}{1-\cos(x)}$ ~~~~ Mult top & bottom by $1+\cos(x)$.

Probability

Feb 26, 2013

Suppose I throw a dart with uniform probability at the interval $[0,1]$. What is the "probability" that I hit a number between:

- 0 and $\frac{1}{2}$?  guess $\frac{1}{2}$
- $\frac{1}{5}$ and $\frac{3}{5}$?  guess $2/5$

Intuitively, the "probability" should be the area of the target, over the area of the whole space.
In general, we need consider weighting - i.e. a "probability density", which measures the probability per unit x.

Def'n: A function $p(x)$ is a probability density function (pdf) on $[a,b]$ if:

1) $p(x) \geq 0$ for x in $[a,b]$
2) $\int_a^b p(x)dx = 1$

eg: The uniform pdf on $[a,b]$ is given by $p(x) = \frac{1}{b-a}$.

The probability that a random variable X takes on values in the interval $a_1 \leq x \leq a_2$ is defined as

$$\int_{a_1}^{a_2} p(x)\,dx$$ ← Think of this as $Pr[a_1 \leq X \leq a_2]$

Note: we will not consider "the probability that X takes on a specific value x_0", since by def'n, this is

$$\int_{x_0}^{x_0} p(x)\,dx = 0$$

Another def'n: The cumulative distribution function (cdf) is defined as

$$F(x) = \int_a^x p(s)\,ds$$ ← Think of this as $Pr[a \leq X \leq x]$
$= Pr[X \leq x]$

Properties

1) By FTC I, $F'(x) = p(x)$.

2) Since $p(x) \geq 0$ (on $[a,b]$), we know $F(x)$ is increasing.

3) $F(a) = \int_a^a p(s)\,ds = 0$

4) $F(b) = \int_a^b p(s)\,ds = 1$ by property 2) of pdfs

5) The probability X takes on a value in the interval $[a_1, a_2]$ is
$$\int_{a_1}^{a_2} p(s)\,ds = \int_{a_1}^{a_2} F'(s)\,ds = F(a_2) - F(a_1) \quad \text{by FTC II}$$

Building pdfs:

It's fairly easy to make a pdf. We know by property 1) of pdfs that $p(x) \geq 0$ for x on $[a,b]$, so start with a function $f(x) \geq 0$ on $[a,b]$. Suppose

$$\int_a^b f(x)\,dx = A > 0 \quad \text{(I use } A \text{ because this is area)}$$

then we can define a corresponding pdf

$$p(x) = \frac{1}{A} f(x),$$

so that property 2) holds. This process is called normalization, and the constant $C = \frac{1}{A}$ in front of f is called the normalization constant

Eg. Start with $f(x) = e^{-x}$ on $[0,2]$ $(e^{-x} \geq 0)$

a) Find the pdf, $p(x)$, of f

b) Find the cdf, $F(x)$.

a) $A = \int_0^2 e^{-x}\,dx = \left(-e^{-x}\right)\Big|_0^2 = 1 - e^{-2}$

$\Rightarrow p(x) = \dfrac{1}{1-e^{-2}} e^{-x}$ is the pdf

b) $F(x) = \int_0^x \frac{1}{1-e^{-2}} e^{-s}\, ds$

$= \frac{1}{1-e^{-2}} \left(-e^{-s}\right)\Big|_0^x$

$= \frac{1-e^{-x}}{1-e^{-2}}$ is the cdf.

Eg. Suppose $\frac{1}{1-e^{-2}} e^{-x}$ is a pdf for $0 \le x \le 2$ describing a random variable X. What is the probability that

a) X is greater than 1?

b) X is less than 1?

a) Probability $= \int_1^2 \frac{1}{1-e^{-2}} e^{-x}\, dx$

$= \frac{1}{1-e^{-2}} \left(-e^{-x}\right)\Big|_1^2 \quad \left(= \frac{1}{1-e^{-2}}\left(-e^{-2} - (-e^{-1})\right)\right)$

$= \frac{e^{-1} - e^{-2}}{1-e^{-2}}$

$= \frac{e^{-1}\cancel{(1-e^{-1})}}{\cancel{(1-e^{-1})}(1+e^{-1})} = \frac{1}{e+1} \approx 0.27$

OR Prob. $= F(2) - F(1)$

$= \frac{(1-e^{-2}) - (1-e^{-1})}{1-e^{-2}}$

$= \frac{e^{-1} - e^{-2}}{1-e^{-2}} = \ldots$ same answer!

b) Prob. $= \int_0^1 \frac{1}{1-e^{-2}} e^{-x}\, dx$

$= \frac{1}{1-e^{-2}} \left(-e^{-x}\right)\Big|_0^1$

$= \frac{1-e^{-1}}{1-e^{-2}} = \frac{1}{1+e^{-1}} \approx 0.73$

Obs: Prob $= 1 - \text{Prob}(X \ge 1)$

OR Prob $= F(1) - \cancel{F(0)}^{0} = \dots$

What kind of info about the random variable X can we get from $p(x)$? We already know the chance that X lies in certain intervals. What about the "expected value" of X? Since $p(x)$ is a density, this is interpreted as the "center" of $p(x)$.
Recall that for a mass density $\rho(x)$, we defined the center of mass as:

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

In the case of a pdf, this is simplified, since $\int_a^b p(x)dx = 1$

Def'n: For a random variable X on $[a,b]$ and its pdf $p(x)$, the mean, or average value, or expected value of X, called $\bar{x}$ (sometimes the mean of p) is given by

$$\bar{x} = \int_a^b x p(x)\,dx$$ ← Think of this as a weighted avg.

Note: This is not the same as the average value of $p(x)$:

$$\bar{p} = \frac{1}{b-a}\int_a^b p(x)\,dx \quad \left(= \frac{1}{b-a} \text{ for pdfs}\right)$$

As in the discrete case, there is more than 1 notion of "average", eg "median". The median of a pdf on $[a,b]$, is given by

$$\int_a^{x_{med}} p(x)\,dx = \int_{x_{med}}^b p(x)\,dx = \frac{1}{2}$$

Note: by defn, $F(x_{med}) = \frac{1}{2}$

Eg. Find the mean and median of $\frac{1}{1-e^{-2}}e^{-x}$.

$$\bar{x} = \int_0^2 x \frac{1}{1-e^{-2}} e^{-x}\,dx$$

$$= \frac{1}{1-e^{-2}} \int_0^2 x e^{-x}\,dx \quad \text{IBP (useful for finding } \bar{x})$$

$$= \frac{1}{1-e^{-2}} \left[-xe^{-x} - e^{-x}\right]\Big|_0^2$$

$$= \frac{1}{1-e^{-2}} \left[1 - 3e^{-2}\right] \approx 0.67$$

$$F(x_{med}) = \frac{1}{2} \Rightarrow \frac{1-e^{-x_{med}}}{1-e^{-2}} = \frac{1}{2}$$

$$2 - 2e^{-x_{med}} = 1 - e^{-2}$$

$$1 + e^{-2} = 2e^{-x_{med}}$$

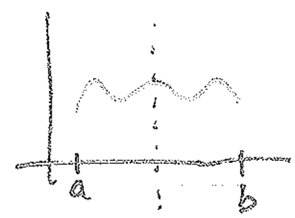
$$\frac{2}{1+e^{-2}} = e^{x_{med}}$$

$$x_{med} = \ln\left(\frac{2}{1+e^{-2}}\right) \approx 0.57$$

Rmk: If $p(x)$ is symmetric about $\frac{a+b}{2}$

We can show

$$\bar{x} = \frac{a+b}{2}.$$



$$\bar{x} = \int_a^b x\, p(x)\,dx \qquad u = x - \left(\frac{a+b}{2}\right)$$

$$= \int_{-\left(\frac{b-a}{2}\right)}^{\frac{b-a}{2}} \left(\underbrace{u}_{\text{odd}} + \left(\frac{a+b}{2}\right)\right) \underbrace{p\left(u + \left(\frac{a+b}{2}\right)\right)}_{\text{even}} du$$

$$= \int_{-\left(\frac{b-a}{2}\right)}^{\frac{b-a}{2}} \left(\frac{a+b}{2}\right) p\left(u + \left(\frac{a+b}{2}\right)\right) du$$

$$= \frac{a+b}{2} \cancelto{1}{\int_a^b p(x)\,dx} = \frac{a+b}{2}$$

$$1 = \int_a^b p(x)\,dx = \int_a^{\frac{a+b}{2}} p(x)\,dx + \underbrace{\int_{\frac{a+b}{2}}^b p(x)\,dx}_{= \text{ by symmetry}} = 2\int_a^{\frac{a+b}{2}} p(x)\,dx$$

So $x_{med} = \frac{a+b}{2}$.

Whenever $p(x)$ is symmetric, mean and median coincide. Otherwise, they might not.

Decay: want a pdf which represents the probability per unit time an atom will decay. It turns out that a good candidate is

$$p(t) = Ce^{-kt}$$

where k is a constant which represents the rate of decay (in units of Hz). In principle, t can be any non-negative. $(t \geq 0)$

We only examined pdfs on finite intervals $[a,b]$.

Now, property 2) becomes

$$2') \int_0^\infty p(t)\,dt = 1$$

<u>Aside:</u> Improper integrals (type I)

If $f(x)$ is continuous for $x \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{s\to\infty}\int_a^s f(x)\,dx$$

In our case, $\int_0^\infty p(t)\,dt = \lim_{s\to\infty}\int_0^s p(t)\,dt \overset{\text{want}}{=} 1$

We need to normalize.

$$\int_0^\infty Ce^{-kt}\,dt = \lim_{s\to\infty}\int_0^s Ce^{-kt}\,dt$$

$$= \lim_{s\to\infty} C\left(-\frac{1}{k}e^{-kt}\right)\Big|_0^s$$

$$= \lim_{s\to\infty} C\left(\frac{1}{k} - \frac{1}{k}e^{-ks}\right)$$

Note $k>0$, so $\lim_{s\to\infty} e^{-ks}=0$, so

$$1=\int_0^\infty Ce^{-kt}\,dt = \lim_{s\to\infty} C\left(\tfrac{1}{k}-\tfrac{1}{k}e^{-ks}\right) = C\frac{1}{k}$$

$\Rightarrow C = k$ ← normalization constant.

The pdf $p(t)=ke^{-kt}$.

Let's find the cdf.

Aside: If $p(x)$ is a pdf on $[a,\infty)$, then the cdf F

$$F(x)=\int_a^x p(s)\,ds$$

Note: as $x\to\infty$, $\int_a^x p(s)\,ds \to \int_a^\infty p(s)\,ds = 1$

so $\lim_{x\to\infty} F(x) = 1$

Might be written: $F(\infty)=1$

<u>exer:</u> Find the cdf for $p(t)=ke^{-kt}$.

Chapter 8: Differential Equations Mar 7/13

Question: What function has derivative equal to x?

Find y such that $\frac{dy}{dx} = x$

$$dy = x\,dx$$
$$\int dy = \int x\,dx$$
$$y = \frac{x^2}{2} + C.$$

Q: Find y such that $\frac{dy}{dx} = xy$ (Later).

Goal: To justify all the differential equations of chapter 4.

$$\frac{dy}{dt} = ky \Rightarrow y = y_0 e^{ky}.$$

Def'n: A separable equation is one of the form
$$\frac{dy}{dx} = f(y)\,g(x).$$

ie. the LHS is a derivative and the RHS is a product of a function of y and a function of x.

The technique to solve these is to bring all the terms depending on y to one side and similarly for x

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Ex: Linear differential equation. Let $a, b \in \mathbb{R}$. Suppose that

$$\frac{dy}{dt} = a - by \quad \text{and} \quad y(0) = y_0 \leftarrow \text{initial condition.}$$

← think $y(t)$ but write only y.

Case 1: $b=0$. Then

$$\frac{dy}{dt} = a$$

$$dy = a\,dt$$

$$\int dy = \int a\,dt$$

$$y = at + C$$

when $t=0$, $y(0)=y_0$ so $y_0 = C$ and hence

$$y(t) = at + y_0$$

Case 2: $b \neq 0$. Separate variables and get

$$\frac{dy}{a-by} = dt$$

Integrating both sides yields

$$\int \frac{dy}{a-by} = \int dt$$

$$-\frac{1}{b}\ln|a-by| = t + C$$

$$\ln|a-by| = -bt - bC$$

$$|a-by| = e^{-bC} e^{-bt} \qquad (\star)$$

When $t=0$, $y(0)=y_0$. So

$$|a-by_0| = e^{-bC} \underbrace{\boxed{e^{-b(0)}}}_{=1}$$

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Plugging into ($*$) gives

$$|a-by| = |a-by_0| e^{-bt}$$
$$a-by = \pm(a-by_0)e^{-bt}$$
$$-by = -a \pm (a-by_0)e^{-bt}$$
$$y = \frac{a}{b} \mp \frac{(a-by_0)}{b} e^{-bt}$$
$$\boxed{y = \frac{a}{b} \pm \left(y_0 - \frac{a}{b}\right) e^{-bt}}$$

Most general version.

when $t=0$, we get $y_0 = \frac{a}{b} \pm \left(y_0 - \frac{a}{b}\right)\boxed{e^{-b(0)}}^{=1}$

need positive sign to get equality.

Thus, $$\boxed{y(t) = \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) e^{-bt}} \quad (1)$$

Let's see how this relates to chapter 4.

(1) If $(a,b) = (0,k)$ where k is another constant. Plug into (1)

$$\frac{dy}{dt} = 0 - ky \Rightarrow y(t) = y_0 e^{-kt}$$

population growth/decay.

(2) If $(a,b) = (g,0) = (9.8,0)$ then by case 1 above, we have

$$\frac{dy}{dt} = g \Rightarrow y(t) = gt + y_0$$

this was our velocity when $y(t) = v(t)$.
$y_0 = v(0)$

(3) If $(a,b) = (g,k)$ with $k \neq 0$, we have

$$\frac{dy}{dt} = g - ky \quad \Rightarrow \quad y = \frac{g}{k} + \left(y_0 - \frac{g}{k}\right)e^{-kt}$$

In our example, $y(t) = v(t)$ $y_0 = v(0) = 0$ so rearranging gives

$$v(t) = \frac{g}{k} - \frac{g}{k}e^{-kt} = \frac{g}{k}\left(1 - e^{-kt}\right)$$

Reminder: Terminal velocity in (3) above (ie when $t \to \infty$) is
$$v(t) \xrightarrow{t\to\infty} \frac{g}{k}$$

This is called a steady state.

Def'n: A steady state of a differential equation is the value where no further change occurs. It can computed by setting the derivative to be 0.

Ex: If $\frac{dv}{dt} = g - kv$, then when $\frac{dv}{dt} = 0$, we get $0 = g - kv$ or $v = \frac{g}{k}$.

Ex: Solve the initial value problem

$$\frac{dy}{dx} = xy \qquad \& \; y(0) = -e$$

Solution: Separating variables gives

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$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C \qquad (\triangle)$$

Plug in our initial value $y(0) = -e$ gives

$$1 = \ln|-e| = \ln|y(0)| = C \qquad \underline{\text{So}} \quad C = \underline{1}.$$

Plug into $(\triangle)$ gives $\ln|y| = \frac{x^2}{2} + 1$

$$|y| = e^{x^2/2 + 1}$$

$$y = \pm e^{x^2/2 + 1}$$

positive exponent.

Plug in the initial condition again gives $y(0) = \pm e$ so since $y(0) = -e$ we want the ~~+~~ negative sign.

Therefore $\boxed{y(t) = -e^{x^2/2 + 1}}$.

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Integrate the following

1. $\int_0^{\pi} (\sin(x)\cos(x))^3 \, dx$

2. $\int \ln(x) \, dx$

3. $\int e^{\sqrt{x}} \, dx$

4. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{x}{\sqrt{1+x^2}} \, dx$

5. $\int x e^x \, dx$

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(1) $\int_0^{\pi} (\sin(x)\cos(x))^3\,dx = \int_0^{\pi} \sin^3(x)\cos^3(x)\,dx$

(1) $= \int_0^{\pi} \sin^2(x)\cos^3(x)\sin(x)\,dx$

(1) $= \int_0^{\pi} (1-\cos^2(x))\cos^3(x)\sin(x)\,dx$ Let $u = \cos(x)$, $du = -\sin(x)\,dx$

1 for changing endpoints
1 for subbing correctly.

$= \int_1^{-1} (1-u^2)u^3(-du)$ $u(0)=1$, $u(\pi) = -1$

$= \int_{-1}^{1} (u^3-u^5)\,du$

(1) $= 0$ since u^3-u^5 is odd & $[-1,1]$ is symmetric about 0.

(2) $\int \ln(x)\,dx$ $u = \ln(x)$, $v = x$, $du = \frac{dx}{x}$, $dv = dx$

3 for parts correctly
2 final answer.

$= x\ln(x) - \int x\frac{dx}{x} = x\ln(x) - x + C.$

-1 if no C.

(3) $\int e^{\sqrt{x}}\,dx$ Let $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2u}$

1 for sub.

$= \int 2ue^u\,du$ $w = u$, $v = e^u$, $dw = du$, $dv = e^u\,du$

2 for parts.

$= 2\left(ue^u - \int e^u\,du\right)$

$= 2ue^u - 2e^u + C.$

1 for this line.

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$

1 for back subbing

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(4) $\int_{\sqrt{3}}^{\sqrt{8}} \frac{x}{\sqrt{1+x^2}}\,dx$ Let ~~u=1+x~~ $u = 1+x^2$

$du = 2x\,dx$

$u(\sqrt{3}) = 4$

$u(\sqrt{8}) = 9$

$\int_4^9 \frac{1}{\sqrt{u}}\left(\frac{du}{2}\right)$ ~~1~~ 2 for sub

1 for change endpoints.

$= \sqrt{u}\Big|_4^9$ 2 for integral

$= \sqrt{9} - \sqrt{4}$

$= 3-2$ 1 for answer.

$= 1.$

(5) $\int x e^x\,dx$ Let $u = x$ $v = e^x$

$du = dx$ $dv = e^x\,dx$

$= xe^x - \int e^x$ 3 for parts

$= xe^x - e^x + C.$ 2 for answer.   -1 if no C.

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Solids of Revolution

arclength

Indefinite Integrals

Trig integrals

Fundamental Theorem of Calculus.

centre ~~[illegible]~~ of mass
centroid.

Substitution

Probability

Variance moments.

pdf
cdf

mean
median
mode.

Partial fractions

By parts

Trig Subs.

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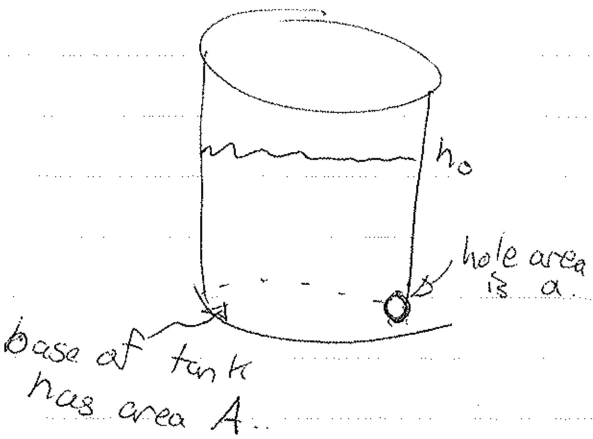
Question: Suppose we have a cylindrical tank filled initially to height h_0. Suppose we put a hole in the tank of area a. Compute the function $h(t)$ ie the height of water left in the tank after time t.

Soln: Assume that gravity is $9.8\,m/s^2$.



$\frac{dV}{dt}$ = negative rate; volume lost as fluid flows out.

$$= -av \;=\; -a\sqrt{2g}\,\sqrt{h} \quad (1)$$

Conservation of energy says

$$\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh}$$

For a cylinder, $V(t) = A\,h(t)$

$$-a\sqrt{2g}\sqrt{h} = \frac{dV}{dt} = A\frac{dh}{dt}$$

from (1)

Solving for $\frac{dh}{dt}$ gives

Call this K

$$\frac{dh}{dt} = \boxed{-\frac{a}{A}\sqrt{2g}}\,\sqrt{h} = -K\sqrt{h}$$

(Toricelli's Law)

This is now a problem of solving a differential equation.

Separating variables gives

$$\frac{dh}{\sqrt{h}} = -K\,dt$$

Integrating gives

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$$\int \frac{dh}{\sqrt{h}} = \int -k\,dt$$

$$2\sqrt{h} = -kt + C$$

Since $h(\underset{t=0}{0}) = h_0$, we see that

$$2\sqrt{h_0} = -k(0) + C = C.$$

So $C = 2\sqrt{h_0}$

Thus,

$$2\sqrt{h} = -kt + 2\sqrt{h_0}$$
$$\sqrt{h} = -\frac{k}{2}t + \sqrt{h_0}$$
$$h(t) = \left(\sqrt{h_0} - \frac{k}{2}t\right)^2$$

Where $k = \frac{a}{A}\sqrt{2g}$ ☒

Follow up: How long does it take to empty?

Answer: The t value where $h(t) = 0$.

So $0 = h(t) = \left(\sqrt{h_0} - \frac{k}{2}t\right)^2 \Rightarrow 0 = \sqrt{h_0} - \frac{k}{2}t$
$\Rightarrow t = \frac{2\sqrt{h_0}}{k}$ ☒

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Logistic Equation

Let's Revisit Population Density.

$$\frac{dy}{dt} = ky \Rightarrow y(t) = y_0 e^{kt}.$$

This model is unrealistic since populations do not grow indefinitely.

Solution: Logistic Equation!

Let $N(t)$ be the size of a population at time t. Let N_0 be the initial population. Then, the logistic differential equation states that the rate of change of population is given by

think $N(t)$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Where $r > 0$ is the "intrinsic growth rate" and $K > 0$ is the carrying capacity.

This K reflects the size of the population that can be sustained by the environment.

~~Let~~ To solve this, let's start by dividing by K.

$$\frac{1}{K}\frac{dN}{dt} = r\frac{N}{K}\left(1 - \frac{N}{K}\right)$$

$$\frac{d(N/K)}{dt} = r\frac{N}{K}\left(1 - \frac{N}{K}\right)$$

Let $y(t) = N(t)/K$. Then, we have

$$\frac{dy}{dt} = ry(1-y) \qquad \& \; y(0) = \frac{N(0)}{K} = \frac{N_0}{K}$$

This process is called "scaling". This new variable $y(t)$ measures the population size in terms of the carrying capacity.

Let's solve the DE.

$$\frac{dy}{y(1-y)} = r\,dt$$

$$\int \frac{dy}{y(1-y)} = \int r\,dt$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{A(1-y)+B(y)}{y(1-y)}$$

$$\Rightarrow \quad 1 = A(1-y) + By$$

$y=0$ gives $1 = A$

$y=1$ gives $1 = B$.

so $$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int r\,dt$$

$$\ln|y| - \ln|1-y| = rt + C$$

$$\ln\left|\frac{y}{1-y}\right| = rt + C$$

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For simplicity, assume that at some point t_0, we have

$$0 \le y(t_0) < 1$$

It will turn out that under ~~as~~ this assumption,

$$0 \le y(t) < 1 \quad \text{for all } t.$$

Exponentiating & removing absolute values since now $\frac{y}{1-y} > 0$, we get

$$\frac{y}{1-y} = e^{(rt+c)} = Be^{rt} \quad \text{where } B = e^c.$$

When $t=0$, $y(0) = N_0/K$ Thus

$$\frac{N_0}{K-N_0} = \frac{N_0/K}{1-N_0/K} = Be^{r(0)} = B$$

Isolating for y yields

$$y = (1-y)Be^{rt}$$
$$y = Be^{rt} - Be^{rt}y$$
$$Be^{rt}y + y = Be^{rt}$$
$$(Be^{rt}+1)y = Be^{rt}$$
$$y = \frac{Be^{rt}}{1+Be^{rt}} = \frac{1}{B^{-1}e^{-rt}+1}$$

Now, $B^{-1} = \frac{K-N_0}{N_0}$ so multiplying top & bottom by N_0 gives

$$y(t) = \frac{N_0}{N_0 B^{-1} e^{-rt} + N_0}$$

$$= \frac{N_0}{(K-N_0)e^{-rt} + N_0}$$

~~OR~~ in terms of y_0

$$= \boxed{\frac{y_0}{(1-y_0)e^{-rt} + y_0}}$$

Let's ~~examine~~ examine

$$y(t) = \frac{y_0}{(1-y_0)e^{-rt} + y_0}$$

Finding the steady state (ie as $t \to \infty$) we see

$$y(t) \xrightarrow{t \to \infty} \frac{y_0}{y_0} = 1.$$

This function is strictly increasing, so we get that $0 \leq y(t) < 1$ for all t as claimed before.

We could also have seen the steady state by setting $\frac{dy}{dt} = 0$ in the original ~~DE~~ differential equation ie

$$0 = \frac{dy}{dt} = ry(1-y) \quad \text{so } y = 0 \text{ OR } y = 1.$$

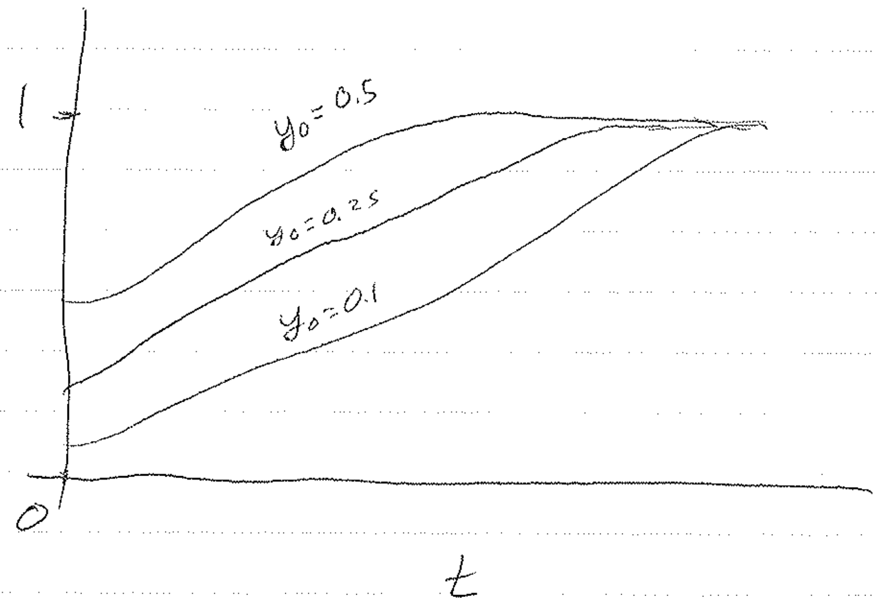
No population
No growth.

~~So~~ Since $y(t) = \frac{N(t)}{K}$, as $t \to \infty$, we know $y(t) \xrightarrow{t \to \infty} 1$

So $N(t) \xrightarrow{t \to \infty} K$.

This justifies the name "carrying capacity" for K.

Samples of $y(t)$



Sequences

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<u>Def'n</u>: A sequence is a list of (real) numbers

$$a_1, a_2, a_3, \dots$$

Sequences can be thought of as functions ~~for~~ from the natural numbers to the real numbers.

$$f : \mathbb{N} \to \mathbb{R} \qquad f(n) = a_n.$$

We write sequences as

$$\{a_1, a_2, a_3, \dots\} \text{ or } \{a_n\} \text{ or } \{a_n\}_{n \geq 1} \text{ or } \{a_n\}_{n=1}^{\infty}$$

We call a_n the n^{th} term of the sequence.

Ex: $\{1, 2, 3, 4, 5, \dots\}$ sequence of natural numbers.

$\{\frac{1}{n^2}\}_{n \geq 1}$ is $1, \frac{1}{4}, \frac{1}{9}, \dots$

$\{\frac{1}{2n+1}\}_{n \geq 1}$ is $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$

$\{1 + \frac{1}{8}, 1 + \frac{1}{16}, 1 + \frac{1}{32}, \dots\}$ is $\{1 + 2^{-n}\}_{n=3}^{\infty}$

$\{2, 3, 5, 7, 11, 13, 17, 19, \dots\}$ is the sequence of prime numbers.

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$ is the Fibonacci sequence

$a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$.

We want to talk about limits.

<u>Ex</u>: Let $a_n = 1 + \frac{1}{n}$.

As n gets large, $\frac{1}{n}$ gets small so a_n tends to $\underline{1}$.

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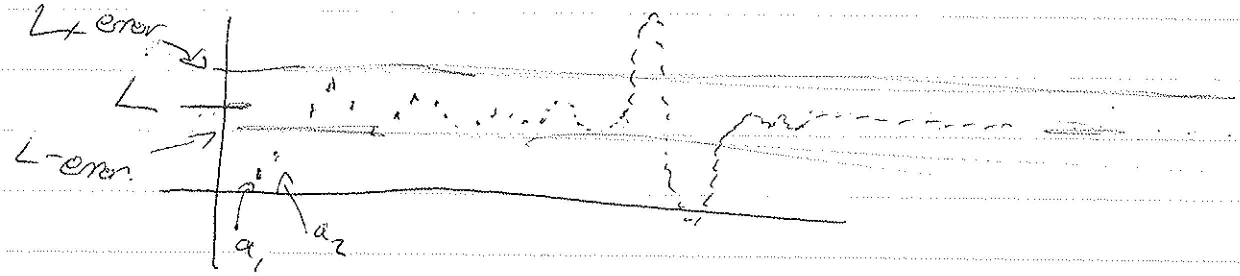
We write $\lim_{n\to\infty} a_n = 1$ OR $a_n \xrightarrow{n\to\infty} 1$.

Def'n: A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{OR} \quad a_n \xrightarrow{n\to\infty} L$$

If we can make the terms a_n as close to L as we like for all n sufficiently large. If limit exists, we say that the sequence is convergent. Otherwise the sequence is divergent.



Q: How are limits of ~~seq~~ sequences and functions related?

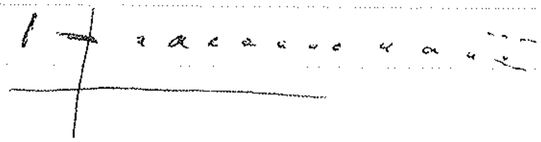
Thm: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Ex: Since $\lim_{x\to\infty} e^{-x} = 0$ Then $e^{-n} \xrightarrow{n\to\infty} 0$.

Let $r > 0$, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$ so $\frac{1}{n^r} \xrightarrow{n\to\infty} 0$.

Ex: If $a_n = 1$ for all $n \geq 1$, does $\{a_n\}_{n\geq1}$ converge?

Ans: Yes! $\lim_{n\to\infty} a_n = 1$. Hence $\{a_n\}_{n\geq1}$ is convergent.



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Properties: Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ for two convergent sequences $\{a_n\}$ & $\{b_n\}$. Let $c \in \mathbb{R}$. Then,

(1) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$.

(2) $\lim_{n\to\infty} c a_n = cA$

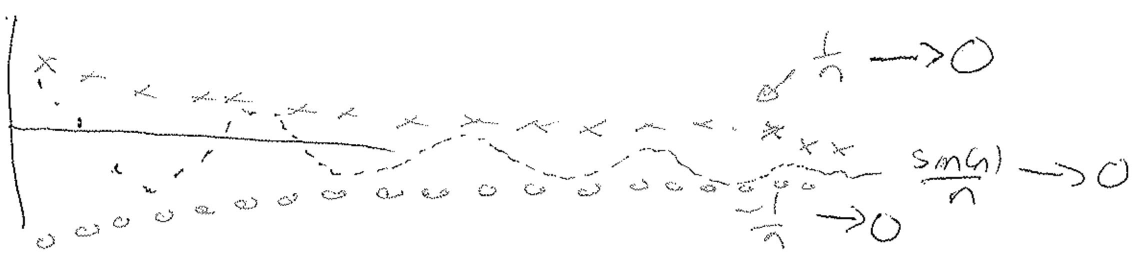
(3) $\lim_{n\to\infty} a_n b_n = AB$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ PROVIDED $B \neq 0$.

(5) $\lim_{n\to\infty} a_n^p = A^p$ if $p>0$ and $a_n > 0$ for all n.

Ex: $$\lim_{n\to\infty} \frac{3n}{2n+7} = \lim_{n\to\infty} \frac{n(3)}{n\left(2+\frac{7}{n}\right)} = \lim_{n\to\infty} \frac{3}{2+\frac{7}{n}} \overset{\text{by (1) \& (4)}}{=} \frac{3}{2+\lim_{n\to\infty}\frac{7}{n}} = \frac{3}{2}.$$

What about $a_n = \frac{\sin(n)}{n}$? Does $\lim_{n\to\infty} \frac{\sin(n)}{n}$ converge?



Thm: (Squeeze Theorem) If $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ are sequences such that

$$a_n \le b_n \le c_n \text{ for all } n \text{ sufficiently large.}$$

and if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$.

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Ex: Compute $\lim_{n\to\infty} \frac{\sin(n)}{n}$ or show it diverges.

Since $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ for all $n \geq 1$

and $\lim_{n\to\infty}\left(\frac{-1}{n}\right) = 0 = \lim_{n\to\infty}\frac{1}{n}$

we have that $\lim_{n\to\infty}\frac{\sin(n)}{n} = 0$ by the squeeze theorem.

What about $a_n = \frac{(-1)^n}{n}$?

<u>Thm</u>: If $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Ex: If $a_n = \frac{(-1)^n}{n}$, then $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0$ Thus, $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$.

What about $a_n = \ln\left(1+\frac{1}{n^2}\right)$?

<u>Thm</u>: If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

In our example, $\lim_{n\to\infty}\ln\left(1+\frac{1}{n^2}\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)\right) = \ln(1) = 0$

by continuity of $\ln(x)$.

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Ex: $a_n = \dfrac{n!}{n^n}$ Reminder $n! = n(n-1)\cdots(3)(2)(1)$

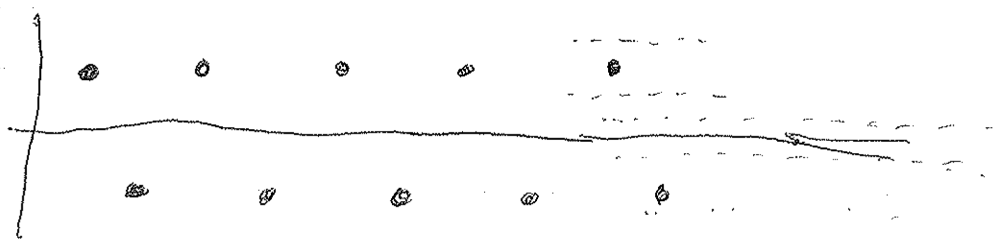
$$0 \le a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdots\cdots\frac{n}{n}\right) \underset{\text{for } n\ge 2}{\le} \frac{1}{n}(1) = \frac{1}{n}.$$

By the squeeze theorem, since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$,

we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

What about divergent sequences?

Look at $(1, -1, 1, -1, 1, -1, \cdots)$



Is the limit 0? No there are no points within $\frac{1}{2}$ of 0.
Is the limit 1? No. Again taking an error of $\frac{1}{2}$ shows that the even numbered terms in the sequence (ie the "-1" terms) are too far away.
Is the limit -1? No, analogous as above.

What goes wrong is that we have two subsequence (sequence inside a sequence) that converge to different limits.

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} 1 = 1 \qquad \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1) = -1.$$

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Theorem: Convergent sequences have unique limits. Therefore, if a sequence has two subsequences that converge to different values, then your original sequence is divergent.

Ex: $\{1, 2, 3, 4, \ldots\} = \{a_n\}$ $\quad \lim_{n\to\infty} a_n$ Does not exist,

thus $\{a_n\}_{n\geq 1}$ is divergent. (As $n \to \infty$, $a_n \to \infty$).

Defn: A sequence $\{a_n\}_{n\geq 1}$ is

- Increasing if $a_n < a_{n+1}$ for all $n \geq 1$
- Non decreasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
- Decreasing if $a_n > a_{n+1}$ for all $n \geq 1$
- Non increasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.

If a sequence is one of these, we say that it is monotonic.

- Bounded above if there is an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$
- Bounded below if there is an $m \in \mathbb{R}$ such that $m \leq a_n$ for all $n \geq 1$
- Bounded if it is bounded above & below.

Ex: $\left\{\frac{1}{n+2}\right\}_{n=1}^{\infty}$ This is bounded above by $\frac{1}{3}$, bounded below by 0 & is decreasing since $\frac{1}{n+2} > \frac{1}{(n+1)+2}$ for all $n \geq 1$.

Does it converge? Yes.

Thm: (Monotone Convergence Theorem)
Every bounded monotonic sequence converges.

This does NOT tell you the limit – only that it exists.

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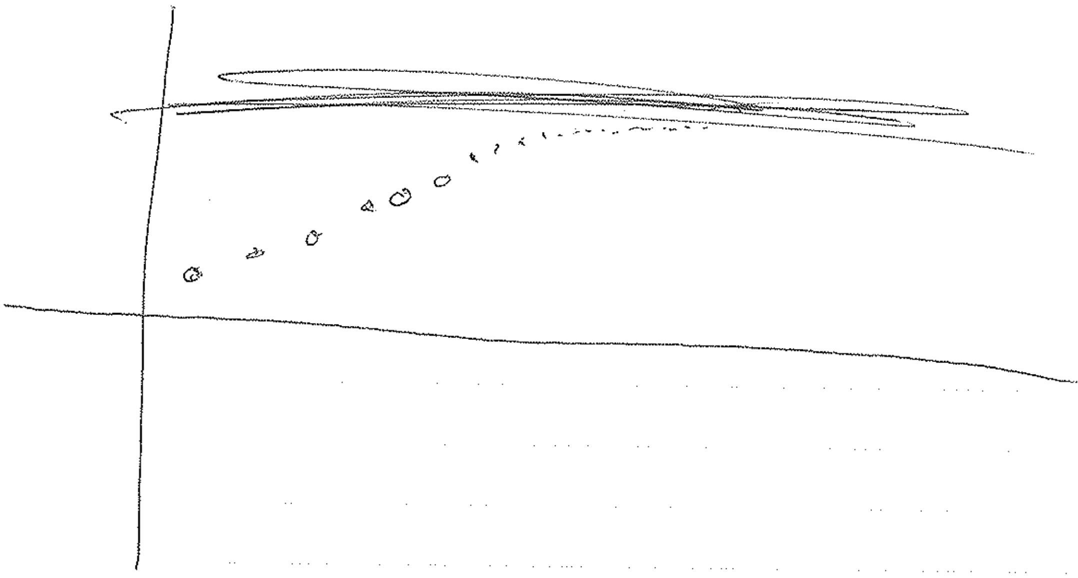

(Chapter 4).

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Theorem:

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r|>1 \text{ or } r=-1 \end{cases}$$

Pf: If $r=1$ + this is clear ✓

If $r=-1$, then this is $(-1, 1, -1, 1, \dots)$ & this diverges.

If $r>1$, this blows up.

if $r<-1$, use previous two cases.

if $-1<r<0$, then look at $|r|$ and note that this will converge by the following

if $0\le r<1$, then look at

Eg: Let $r=\frac{1}{2}$, then $\left(\frac{1}{2}\right)^n \xrightarrow{n\to\infty} 0$

Similar reasoning shows $r^n \xrightarrow{n\to\infty} 0$ when $0\le r<1$. ∎

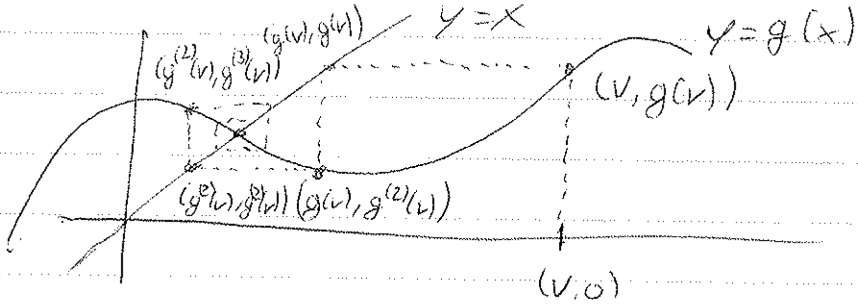
Iterated maps.

Let $g(x)$ be a function. Denote by

$g^{(1)}(x) = g(x)$ $\qquad g^{(2)}(x) = g(g(x))$.

an in general, let $g^{(k)}(x) = \underbrace{g(g\cdots}_{k \text{ times}} g(x))\cdots)$ be the k^{th} iteration

Let's discuss cobwebbing



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(1) Start with $(v,0)$ for some v
(2) Draw a vertical line to $(v, g(v))$ (ie to the function).
(3) Draw a horizontal line to $(g(v), g(v))$ (ie to the line $y=x$).
(4) Draw a vertical line to $(g(v), g^{(2)}(v))$ (ie to the function.
(5) Repeat 3 & 4.

Suppose further that $g(x)$ is continuous and that

$$(a_k)_{k\geq 0} = (v, g(v), g^{(2)}(v), \ldots)$$

converges say $\lim_{k\to\infty} a_k = \lim_{k\to\infty} g^{(k)}(v) = V$

The notice that

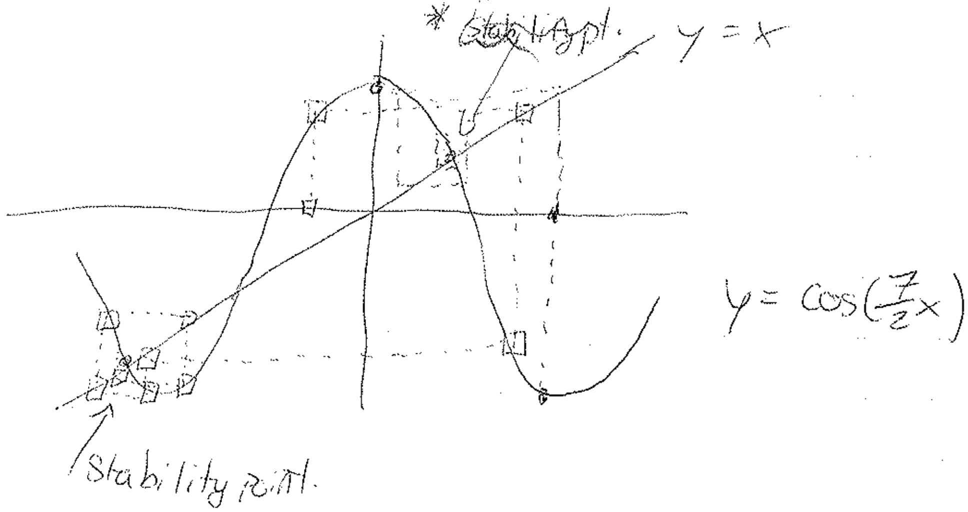
$$g(V) = g\left(\lim_{k\to\infty} g^{(k)}(v)\right) \overset{\text{by continuity}}{=} \lim_{k\to\infty} g(g^{(k)}(v)) = \lim_{k\to\infty} g^{(k+1)}(v) = V$$

ie V is a fixed point of g, that is, $g(V)$ lies on $y=x$

When this happens, we call V a stability point.

NB: Not all fixed points are stability points!

Ex:



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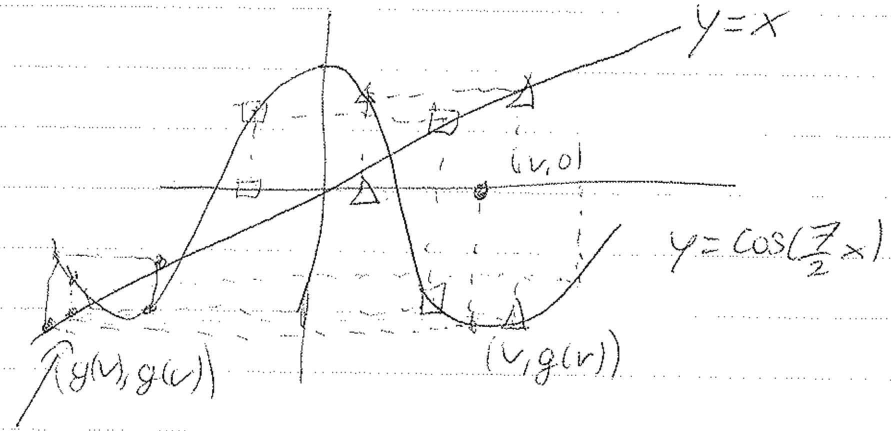


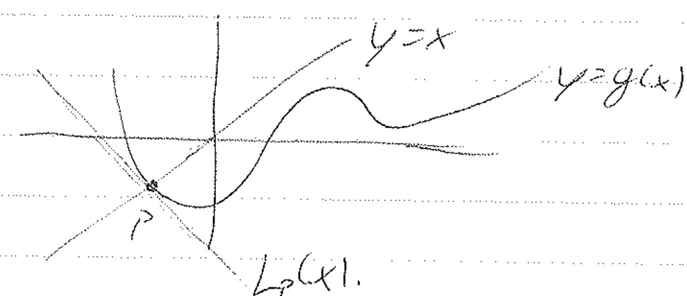
only stability point! Even though $g(x)$ intersects $y=x$ 3 times.

Iterated map sequences that do not converge are called unstable.

Q: When is a fixed point stable?

It turns out that this depends on the angle at which $g(x)$ intersects the line $y=x$.

Let p be a fixed point so $g(p)=p$. Let $L_p(x) = g'(p)x + p(1-g'(p))$ be the equation of the tangent line at p. (Note: we also require now that $g(x)$ is differentiable) (double check that $L_p(x)$ is correct.)



Suppose that our starting point x is "close to p". Whenever $L_p(x)$ is closer to p then x is, we get a stability point. Let's look at

i.e. ~~what~~ $|L_p(x) - p| < |x-p|$

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This occurs when

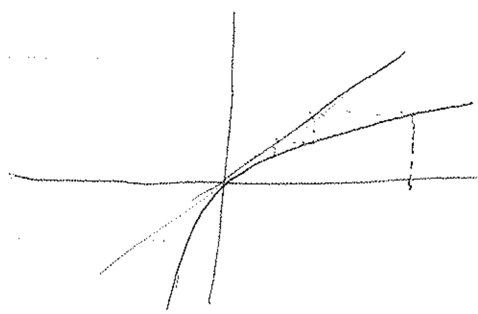
Recall: $L_p(x) = g'(p)x + p(1-g'(p))$,

$$|L_p(x) - p| = |g'(p)x + p(1-g'(p)) - p| = |g'(p)||(x-p)|$$

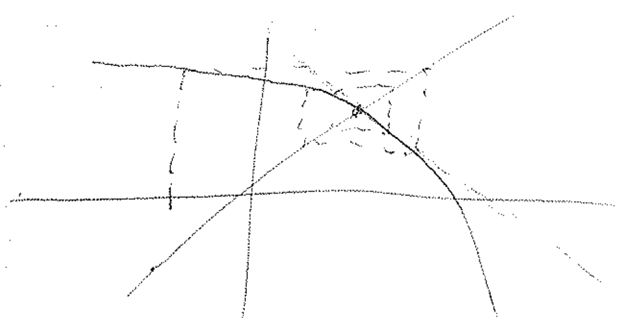
This holds when $|g'(p)| < 1$. Rewording, if $|g'(p)|<1$, then p is a stability point.

A fixed point p is a stability point provided $|g'(p)|<1$.

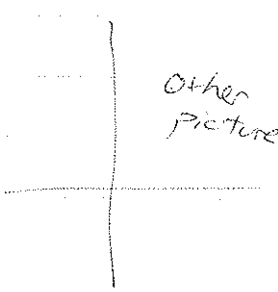
If $0 < g'(p) < 1$, the cobweb becomes a staircase.

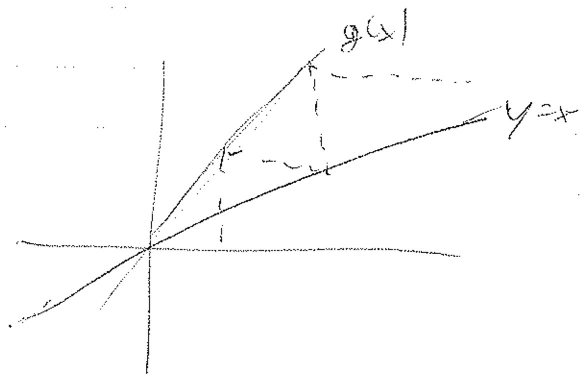


If $-1 < g'(p) < 0$, we get a spiral



Otherwise, we get instability.





9.2 Difference Equation

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Let N_t be the # of individuals at time t.
Let α be the average offspring produced per unit ~~~~ time t.

$$N_{t+1} = \alpha N_t$$

If we start with N_0 people, then

$$N_1 = \alpha N_0, \quad N_2 = \alpha N_1 = \alpha^2 N_0, \ldots \qquad N_t = \alpha^t N_0.$$

Let's relate to & differential equations.

Subtract N_t from both sides,

$$N_{t+1} - N_t = (\alpha - 1) N_t$$

Write N_t as $N(t)$ ie think of N_t as a function and divide by Δt.
and replace 1 with Δt

unit of time.

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = \left(\frac{\alpha^{\Delta t} - 1}{\Delta t}\right) N(t)$$

Take $\lim_{\Delta t \to 0}$ on both sides.

$$\frac{dN}{dt} = \ln(\alpha)\, N(t)$$ like our old population model!

Solving this DE gives $N(t) = N_0 \alpha^t$.

NB. $\alpha > 1$, population blowsup, if $\alpha < 1$, population dies.
$\alpha = 1$ constant.

Logistic Map. Mar 14/13

Now, introduce γ = strength of competition. We change our model to

$$N_{t+1} = \alpha N_t - \gamma N_t^2.$$

Now, per time step,

$$\frac{N_{t+1}}{N_t} = \alpha - \gamma N_t$$

Similar to the logistic equation

$$\frac{dN}{dt} = rN - \frac{r}{K}N^2 = rN\left(1 - \frac{N}{K}\right)$$

Where $r = \alpha$ and $K = \alpha/\gamma$.

As before, we want to check when $\alpha - \gamma N_t$ is bigger than, less than or equal to 1 to determine population pattern.

Suppose $\gamma > 0$.

So if $\alpha - \gamma N_t < 1$, then $\alpha - 1 < \gamma N_t \Rightarrow \frac{\alpha - 1}{\gamma} < N_t$.

Thus, when $N_t > \frac{\alpha - 1}{\gamma}$, our population size declines.

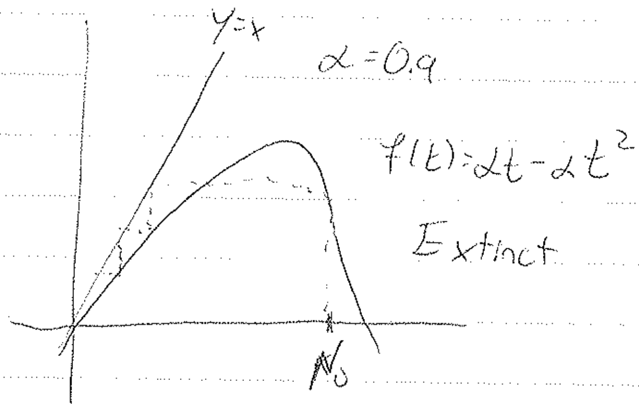
Similarly if $\alpha - \gamma N_t > 1 \Rightarrow \frac{\alpha - 1}{\gamma} > N_t \Rightarrow$ our population is increasing.

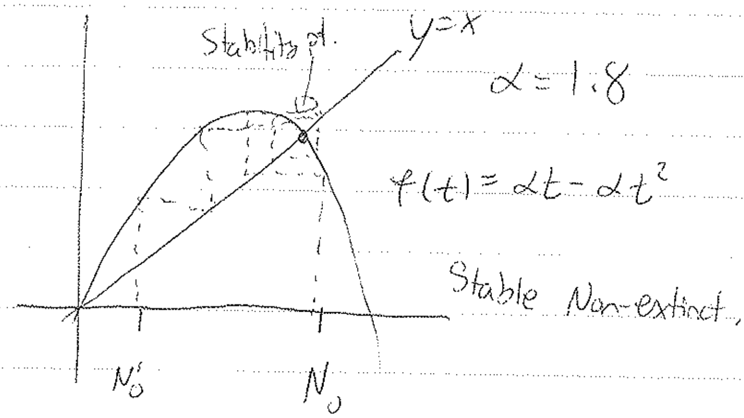
Lastly, if $\alpha - \gamma N_t = 1 \Rightarrow \frac{\alpha - 1}{\gamma} = N_t \Rightarrow$ population remains constant.

We sometimes call $\frac{\alpha - 1}{\gamma} = N^*$ the carrying capacity.

recurrence equation

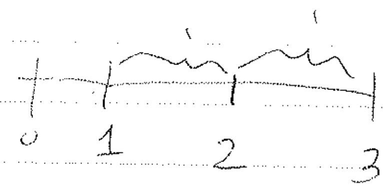
Ex: $N_{t+1} = \alpha N_t (1-N_t) = \alpha N_t - \alpha N_t^2$





Stability at fixed point $x^* = 1-\frac{1}{\alpha}$ when

$$|f'(x^*)| = |\alpha - 2\alpha x^*| < 1$$
$$|\alpha - 2\alpha(1-\tfrac{1}{\alpha})| < 1$$
$$|\alpha - 2\alpha + 2| < 1$$
$$|2-\alpha| < 1$$



$$\Rightarrow 1 < \alpha < 3.$$

So when $1 < \alpha < 3$, we have a fixed point $x^* = 1 - \frac{1}{\alpha}$.

Improper Integrals (Chapter 10) Mar-19/18

Now, we formally treat ~~two~~ the two kinds of improper integrals.

Improper Integrals of the First Kind.

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

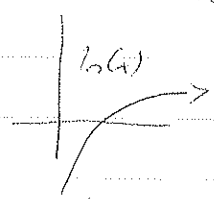
If the limit on the right exists, we say the improper integral converges and we sometimes denote this as

$$\int_a^\infty f(x)\,dx < \infty$$

otherwise the integral diverges.

Ex: $$\int_1^\infty e^{-t}\,dt = \lim_{b\to\infty}\int_1^b e^{-t}\,dt = \lim_{b\to\infty} -e^{-t}\Big|_1^b$$
$$= \lim_{b\to\infty} -e^{-b} - (-e^{-1}) = 0 + \frac{1}{e} = \frac{1}{e}$$

Ex: $$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty}(\ln(b) - \ln(1))$$
$$= \cancel{DNE}\ \infty.$$



Thus, $\int_1^\infty \frac{dx}{x}$ diverges.

Ex: $$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -x^{-1}\Big|_1^b = \lim_{b\to\infty}\left[-(b)^{-1} - (-1)\right]$$
$$= 1$$

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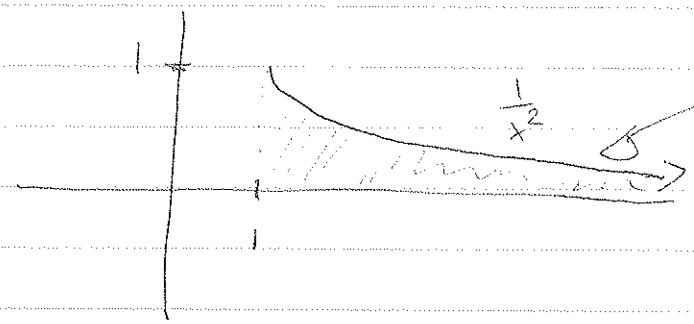
<u>Note</u>:



finite area.
infinite "perimeter"

Q. When does $\int_1^{\infty} x^p \, dx$ converge?

Ans: when $p = -1$ we showed above that this integral diverged.
When $p \neq -1$, we have $(p \in \mathbb{R})$.

$$\int_1^{\infty} x^p \, dx = \lim_{b \to \infty} \int_1^b x^p \, dx = \lim_{b \to \infty} \left. \frac{x^{p+1}}{p+1} \right|_1^b = \lim_{b \to \infty} \frac{b^{p+1}}{p+1} - \frac{1}{p+1}$$

This last limit exists when $p+1 < 0$ ie $p < -1$. When $p+1 > 0$ this integral is divergent.

Thus, $\int_1^{\infty} x^p \, dx$ converges to $\frac{-1}{p+1}$ if $p < -1$ and diverges otherwise.
$p \geq -1$

$\int_1^{\infty} \frac{1}{x^q} \, dx$ converges if $q > 1$
diverges if $q \leq 1$

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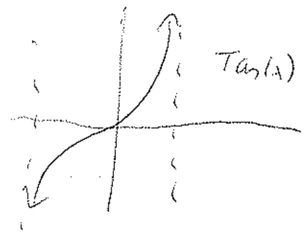
Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

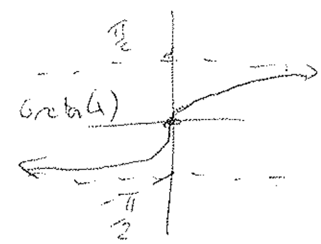
Break this into two integrals using 0.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan(x)\Big|_0^b = \lim_{b\to\infty} \arctan(b) - \arctan(0)$$

$$= \frac{\pi}{2}$$





Similarly,

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \arctan(x)\Big|_a^0 = \lim_{a\to-\infty} \left(\arctan(0) - \arctan(a)\right)$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Combining: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Recall: $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$ does not work since we have a singularity at 0.

How do we deal with these integrals?

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Def'n: Improper integral of the second kind.

- If f is continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$ provided this limit exists and is finite.

- If f is continuous on $(a,b]$ and discontinuous at a. Then
$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$ provided this limit exists and is finite.

- If the limit(s) exist, the integral(s) are said to be convergent. Otherwise, they are divergent.

- If $f(x)$ has a discontinuity at $c\in(a,b)$ and both
$$\int_a^c f(x)\,dx \quad\text{and}\quad \int_c^b f(x)\,dx$$
are convergent, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex. $\displaystyle\int_2^6 \frac{dx}{\sqrt{x-2}}$ ← problem at 2.

$$\int_2^6 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+}\int_t^6 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big|_t^6 = \lim_{t\to 2^+} \overbrace{2\sqrt{6-2}}^{4} - \overbrace{2\sqrt{t-2}}^{0}$$
$$= 4 - 0 = 4.$$

Ex: $$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^2} + \lim_{s\to 0^+}\int_s^1 \frac{dx}{x^2}$$
$$= \lim_{t\to 0^-} -x^{-1}\Big|_{-1}^t + \lim_{s\to 0^+} -x^{-1}\Big|_s^1$$

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$$\int_{-1}^{1} \frac{dx}{x^2} = \underbrace{\lim_{t \to 0^-} -\frac{1}{t} - \left(\frac{-1}{(-1)}\right)}_{-\infty} + \underbrace{\lim_{s \to 0^+} \frac{-1}{1} - \left(-\frac{1}{s}\right)}_{\infty}$$

<u>DIVERGES</u>!

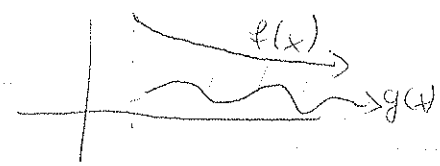
Thus, $\int_{-1}^{1} \frac{dx}{x^2}$ diverges.

The Comparison Theorem.

Suppose f and g be continuous with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent and

$$\int_a^\infty f(x)\,dx \geq \int_a^\infty g(x)\,dx$$



(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Ex: Show $\int_1^\infty e^{-x^2}\,dx$ is convergent.

Plan: Since $\int_1^\infty e^{-t}\,dt = \frac{1}{e} < \infty$ we would like to show that $e^{-x^2} \leq e^{-x}$ between 1 and ∞ and then use the comparison theorem

In order for $e^{-x^2} \leq e^{-x}$, we need $-x^2 \leq -x$ for all $x \in [1, \infty)$

This is true if $x \leq x^2$ for $x \in [1, \infty)$ and this is true!

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Model solution:

Notice that $x \leq x^2$ on $[1, \infty)$ so $-x^2 \leq -x$ on $[1, \infty)$ thus $e^{-x^2} \leq e^{-x}$ for all $x \in [1, \infty)$. Since

$$\int_1^\infty e^{-t}\,dt \text{ converges,}$$

we have that $$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx < \infty$$

Thus, the convergence theorem says $\int_1^\infty e^{-x^2}\,dx$ converges.

Back to Chapter 9. Series. Mar. 19/13

Consider a sequence $\{a_n\}_{n=1}^{\infty}$. We wish to study

$$\sum_{n=1}^{\infty} a_n.$$

What does ~~it~~ this mean.

Ex: $\sum_{n=1}^{\infty} n = 1+2+3+4+\cdots$

Take a look at partial sums

$$S_1 = 1$$
$$S_2 = 1+2 = 3$$
$$S_3 = 1+2+3 = 6$$
$$\vdots$$
$$S_N = 1+2+\cdots+N = \frac{N(N+1)}{2}.$$

As $N \to \infty$, $S_N \to \infty$ so the partial sums diverge and so $\sum_{n=1}^{\infty} n$ doesn't make sense.

Sometimes it is not clear if a series converges or diverges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Recall: Defn: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and let

$$S_N = a_1 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

denote the N^{th} partial sum. If $S_N \xrightarrow{N\to\infty} S$ for some $S \in \mathbb{R}$, then we write $\sum_{n=1}^{\infty} a_n = S$ and say that the series is convergent. If $\{S_n\}_{n=1}^{\infty}$ diverges, then we say that the series diverges.

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Properties of series.

Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ be two convergent series. Let $c \in \mathbb{R}$. Then

$$\left.\begin{aligned} \sum_{n=1}^{\infty} c\,a_n &= cA \\ \sum_{n=1}^{\infty} (a_n \pm b_n) &= A \pm B \end{aligned}\right\} \text{linearity.}$$

NOTE: $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ are not easy to compute.

In general, series are difficult to compute, but there are some cases where we can evaluate them.

Ex: Geometric Series.

$$\text{Ex: } 9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \cdots = \sum_{n=1}^{\infty} 9\left(\frac{-3}{5}\right)^{n-1} = \frac{9}{1-\left(\frac{-3}{5}\right)} = \frac{45}{8}$$

$\frac{a}{1-r}$

Ex! Telescoping Sums. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\text{Partial sums} \left\{\begin{aligned} S_1 &= \tfrac{1}{2} \\ S_2 &= \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{4}{6} = \tfrac{2}{3} \\ S_3 &= \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{9}{12} = \tfrac{3}{4} \end{aligned}\right.$$

$$S_n = \frac{n}{n+1}.$$ Why?

Idea: Partial Fractions! $\frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

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Thus: $S_n = (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}) + \cdots + (\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1})$

$= 1 - \frac{1}{n+1}$

$= \frac{n}{n+1}$.

So $S_n = 1 - \frac{1}{n+1}$. as $n \to \infty$, $S_n \to 1$

Thus, by definition, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Thm: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent

DIVERGENCE TEST.

Ex: Does $\sum_{n=1}^{\infty} \frac{n+2}{2n+3}$ converge or diverge?

Soln: Since $\lim_{n\to\infty} \frac{n+2}{2n+3} = \lim_{n\to\infty} \frac{\cancel{n}(1+\overset{0}{\frac{2}{n}})}{\cancel{n}(2+\underset{0}{\frac{3}{n}})} = \frac{1}{2} \neq 0$.

Thus, the divergence test shows that $\sum_{n=1}^{\infty} \frac{n+2}{2n+3}$ diverges.

NB The converse is FALSE!

think $\sum_{n=1}^{\infty} a_n$ think a_n

There exist ~~seq~~ series with terms that go to zero BUT the series diverges.

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Ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ (Harmonic Series).

Let's take partial sums (the $1^{st}, 2^{nd}, 4^{th}, 8^{th}, \ldots$)

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$
$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$
$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2}\ \frac{1}{2}$$
$$= 2.5$$

$$S_{16} > 3$$
$$\vdots$$
$$S_{2^n} > 1 + \frac{n}{2}.$$

As $n \to \infty$, $(1 + \frac{n}{2}) \xrightarrow{n \to \infty} \infty$ ie $\{1 + \frac{n}{2}\}_{n=1}^{\infty}$ diverges.
Thus, $\{S_{2^n}\}_{n=1}^{\infty}$ diverges! So the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge!!!

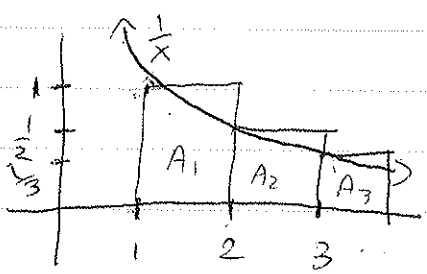
EVEN THOUGH $\lim_{n \to \infty} \frac{1}{n} = 0$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

Our goal now is to determine when a series converges and when it diverges.

Integral Test.

Q: How fast does $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge?

Use left Riemann Sums



$A_1 = 1$ $A_2 = \frac{1}{2}$ $A_3 = \frac{1}{3}$

Maybe, $\sum_{n=1}^{\infty} \frac{1}{n} \approx \int_1^{\infty} \frac{1}{x}\,dx$

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We already saw that $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} \ln(b)$

Combining gives $\sum_{n=1}^{\infty} \frac{1}{n} \approx \lim_{b\to\infty} \ln(b)$

<u>Translation</u>: $\sum_{n=1}^{\infty} \frac{1}{n}$ grows logarithmically!

This is close to the truth! Define

$$\gamma := \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = 0.5772156\ldots$$

(Euler - Mascheroni constant).

Open Problem: Is γ rational?

Thm: (Integral Test).
Suppose that f is a continuous positive decreasing function on $[1,\infty)$. Let $a_n = f(n)$ ~~Then~~ for $n = 1, 2, \ldots$ Then

(a) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n\geq 1} a_n$ is convergent.

(b) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n\geq 1} a_n$ is divergent.

Ex: Since $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} (b)$ ie is divergent, then $\sum_{n\geq 1}^{\infty} \frac{1}{n}$ is divergent.

Ex: Since $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -x^{-1}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{b} + 1\right) = 1$

converges, we have that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

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huh?

Q: What does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ equal? Answer: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Ex: For what p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Note when $p<0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$, thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ will diverge by the Divergence test.

If $p=0$, then $\frac{1}{n^p} = 1$ for all n. So $\lim_{n\to\infty} \frac{1}{n^p} = 1$ & so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the divergence test.

When $p>0$, notice that $\frac{1}{x^p}$ is continuous on $[1,\infty)$, is decreasing and is positive, so we can apply the integral test. Note that

we did this!

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p\le 1 \end{cases}$$

This gives us the p-series test.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\le 1$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^{9973}}$ this converges by the p test (since $9973>1$)

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Let's try $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ (Does this series converge or diverge).

Try integral test. Notice that $\frac{\ln x}{x}$ is continuous, positive and is decreasing.

So by the integral test, since

$u = \ln x \quad u(1) = \ln(1) = 0$
$du = \frac{dx}{x} \quad u(b) = \ln(b)$

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_0^{\ln(b)} u\,du$$

$$= \lim_{b\to\infty} \frac{u^2}{2}\Big|_0^{\ln(b)} = \lim_{b\to\infty} \frac{\ln(b)^2}{2} - 0$$

DIVERGE!

diverges, we have that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

$$\frac{1}{n} < \frac{\ln(n)}{n} \quad \text{for } n \geq 3$$

So we should expect $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ should diverge. We'll make this precise in 4 days.

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Last time, we used the integral test to show that

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges.}$$

Note that $\frac{1}{n} < \frac{\ln(n)}{n}$ for $n \geq 3$.

So we should expect that since

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges then } \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges}$$

just like for integrals. This is true in general.

Theorem: <u>Comparison Test</u>.

Suppose $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ are series with all positive terms. Suppose that $a_n \leq b_n$ for all large values of n. Then

(1) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Ex: Since $\frac{1}{n} < \frac{\ln(n)}{n}$ for $n \geq 3$ and $\sum_{n=1} \frac{1}{n}$ diverges by the p-test, we have that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by the comparison test.

Solution to "Does $\sum_{n=1} \frac{\ln(n)}{n}$ converge or diverge".

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Ex: $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ Intuition: top is n bottom is n^3 so $\frac{\text{top}}{\text{bottom}} \approx \frac{1}{n^2}$

So this should converge. Want a bigger denominator.

Notice that $2n^3 \leq 2n^3+1$ for all $n \geq 1$ so $\frac{1}{2n^3+1} \leq \frac{1}{2n^3} \Rightarrow \frac{n}{2n^3+1} \leq \frac{n}{2n^3} \leq \frac{1}{2n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges by the p-series test, we have that

$\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges by the comparison test.

Ex: $\sum_{n=1}^{\infty} \frac{n+5}{n^{4/3}}$ Intuition: $\frac{n+5}{n^{4/3}} \approx \frac{n}{n^{4/3}} = \frac{1}{n^{1/3}}$ & $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges by the p-test

Notice that $n < n+5$ for all $n \geq 1$. so we have

$$\frac{1}{n^{1/3}} = \frac{n}{n^{4/3}} < \frac{n+5}{n^{4/3}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges by the p-series test

Then $\sum_{n=1}^{\infty} \frac{n+5}{n^{4/3}}$ diverges by the comparison test.

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Last Test: Ratio Test

Def'n: A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm: If a series is absolutely convergent, then it is convergent.

Idea of Ratio test: If adjacent terms get progressively bigger/smaller then the series should diverge/converge.

Thm: Ratio test Let $\sum_{n=1}^{\infty} a_n$ be a series.

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (hence convergent)

(2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ OR $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series diverges.

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the ratio test is inconclusive.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n} \leftarrow a_n$

By the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{7^{n+1}} \div \frac{(-1)^n n}{7^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{7^{n+1}} \cdot \frac{7^n}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(1+\frac{1}{n})}{7} \right| = \frac{1}{7} < 1$$ Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n n}{7^n}$ converges.

<u>Ex:</u> Determine the values of x such that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} \frac{(n+1)}{(n+2)} \quad a_n$$

Sol'n: By the ratio test,

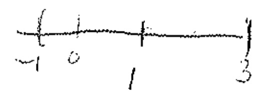
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}} \frac{(n+2)}{(n+3)} \div \frac{(x-1)^n}{2^n} \frac{(n+1)}{(n+2)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)\,(n+2)^2}{2\,(n+3)(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{x-1}{2}\right) \frac{n^2(1+\frac{2}{n})^2}{n^2(1+\frac{3}{n})(1+\frac{1}{n})} \right|$$

$$= \frac{|x-1|}{2}$$

When $\frac{|x-1|}{2} < 1 \implies |x-1| < 2$
$\implies -1 < x < 3$. we have convergence by the ratio test.



When $\frac{|x-1|}{2} > 1$ so $x < -1$ or $x > 3$, our series diverges by the ratio test.

When: $|x-1| = 2$ so $x = -1, 3$ we don't know yet.

When $x = 3$, our series is

$$\sum_{n=1}^{\infty} \frac{(3-1)^n}{2^n} \frac{(n+1)}{(n+2)} = \sum_{n=1}^{\infty} \frac{(n+1)}{(n+2)}$$

Since $\lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1$, the series diverges by the divergence test.

When $x=-1$

$$\sum_{n=1}^{\infty} \frac{(-1-1)^n}{2^n} \frac{(n+1)}{(n+2)} = \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n+2}.$$

Since $\lim_{n\to\infty} (-1)^n \frac{(n+1)}{(n+2)}$ doesnotexist (there isasubsequence ie the even n terms; that goes to 1 and asubsequence that goesto -1, namely the odd numbered terms) the divergence test tells us that our series when $x=-1$ diverges.

So $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} \cdot \frac{n+1}{n+2}$ converges if andonly if $-1 < x < 3$.

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Sample Problems

Determine if the following converge or diverge

(i) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

(ii) $\sum_{n=1}^{\infty} \arctan(n)$

(iii) $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

(v) $\sum_{n=3}^{\infty} \frac{n!}{2^n}$

(vi) $\sum_{n=3}^{\infty} \frac{n^3-1}{n^3+1}$.

(vii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

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Solutions

(1) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ Let $f(x) = \frac{1}{x\ln(x)}$.

Notice that $f(x)$ is continuous on $[2,\infty)$. $f(x)$ is decreasing since if $x<t$ for $x, t \in [2,\infty)$, then as $x\ln(x) < t\ln(t)$ we have that $\frac{1}{t\ln(t)} < \frac{1}{x\ln(x)}$ so $f(x)$ is decreasing. As both x and $\ln(x)$ are positive, we have that $f(x)$ is positive on $[2,\infty)$.

Thus, we can apply the integral test

$$\int_2^\infty \frac{dx}{x\ln(x)} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln(x)}$$

$$= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{du}{u}$$

Let $u = \ln x$ $u(2) = \ln(2)$

$du = \frac{dx}{x}$ $u(b) = \ln(b)$

$$= \lim_{b\to\infty} \ln|u| \Big|_{\ln(2)}^{\ln(b)}$$

$$= \lim_{b\to\infty} \left(\ln(\ln(b)) - \ln(\ln(2))\right)$$

diverges

Since $\int_2^\infty \frac{dx}{x\ln(x)}$ diverges, so does $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$.

(ii) $\sum_{n=1}^{\infty} \arctan(n)$

Since $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$, the series diverges by the divergence test.

(iii) $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ Intuition: numerator $\approx n^2$, denominator $\approx n^4$. So series should be like $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Use the comparison test. Our intuition suggests that this converges so we want an upper bound [a simple one] for n^2-1 and a lower bound for $3n^4+1$.

Sol'n:

For all $n \geq 1$, notice that

$$n^2 - 1 \leq n^2$$
$$3n^4 \leq 3n^4+1 \Rightarrow \frac{1}{3n^4+1} \leq \frac{1}{3n^4}$$

Combining shows

$$\frac{n^2-1}{3n^4+1} \leq \frac{n^2}{3n^4} = \frac{1}{3} \cdot \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ converges by the p-series test, we have that $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges by the comparison test.

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(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Sol'n: Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the p-series test and $\left|\frac{(-1)^n}{n^4}\right| = \frac{1}{n^4}$, we have that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges by the absolute convergence test.

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(v) $\displaystyle\sum_{n=3}^{\infty} \frac{n!}{2^n}$ Intuition: Factorials & exponentials suggest ratio test.

Sol'n Use ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left| \frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n} \right|$$

$$= \lim_{n\to\infty}\left| \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} \right|$$

$$= \lim_{n\to\infty}\left| \frac{(n+1)}{2} \right|$$

$$= \infty$$

So the ratio test tells us that this series diverges.

(vi) $\displaystyle\sum_{n=3}^{\infty} \frac{n^3-1}{n^3+1}$

Since $\displaystyle\lim_{n\to\infty} \frac{n^3-1}{n^3+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n^3}}{1+\frac{1}{n^3}} = 1 \neq 0$, the divergence test shows that this series diverges.

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(viii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ Intuition: numerator ≈ 1 so should be like $\sum_{n=1}^{\infty} \frac{1}{n}$

denominator $\approx \sqrt{n^2} = n$

Sol'n: By our intuition, we want to show divergence. Thus, we need an upper bound on $\sqrt{n^2+1}$ that looks like n.

Notice that

$$n^2+1 \le 4n^2 \quad \text{for all } n \ge 1$$

So $$\sqrt{n^2+1} \le \sqrt{4n^2} = 2n$$

Thus, $$\frac{1}{2n} \le \frac{1}{\sqrt{n^2+1}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by the p-series test, we have that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges by the comparison test.

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Ex: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Let $a_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \div \frac{n!}{n^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)n!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}\right|$$

$(n+1)! = (n+1)n!$
$4! = 4 \cdot 3 \cdot 2 \cdot 1$
$= 4 \cdot 3!$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} e^{\ln\left(\left(\frac{n}{n+1}\right)^n\right)}$$

$e^{\ln x} = x$

$$= \lim_{n\to\infty} e^{\ln\left(\left(\frac{n+1}{n}\right)^{-n}\right)}$$

$$= \lim_{n\to\infty} e^{-n\ln\left(1+\frac{1}{n}\right)}$$

$\frac{1}{\left(\frac{1}{n}\right)} = 1 \div \frac{1}{n} = n.$

$$= e^{-\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$

(valid since e^x is continuous).

$$= e^{-\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}}$$

Let $m = \frac{1}{n}$ as $n\to\infty$
$m \to 0$

$$= e^{-\lim_{m\to 0} \frac{\ln(1+m)}{m}}$$

$$= e^{-\lim_{m\to 0} \frac{\frac{1}{m+1}}{1}}$$

$\leftarrow \frac{d}{dm}(\ln(1+m))$
$\leftarrow \frac{d}{dm}(m)$

L'Hopital's Rule.

$$= e^{-1}$$

$$= \frac{1}{e} < 1$$

So the ratio test says $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

Chapter 11: Taylor Series. Mar. 28/13

Representation of functions as power series.

Recall: $\sum_{n=0}^{\infty} x^n$ ← This is a geometric series. Converges when $|x|<1$.

We had $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ Valid when $|x|<1$

Similarly, we can do

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^n$$

Valid when $|-2x|<1$ so $|x|<\frac{1}{2}$.

A bit more work shows

$$\frac{2x^5}{3-x} = x^5\left(\frac{2/3}{1-\frac{x}{3}}\right) = \frac{2}{3}x^5\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{2x^5}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2x^{n+5}}{3^{n+1}}$$

valid when $|\frac{x}{3}|<1$ so $|x|<3$.

In regions of absolute convergence, we can take integrals & derivatives of power series.

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Theorem: If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on an open interval I, then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is differentiable (hence continuous, hence integrable) on I and

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ ← NOTE NOT 0.

(b) $\int f(x) = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1}$.

All of these powerseries still converge on I.
For us, $a=0$ almost always.

Ex: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. for $x \in (-1,1)$ (ie $|x|<1$).

$$\underbrace{\frac{d}{dx} \frac{1}{1-x}}_{= \frac{1}{(1-x)^2}} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$ ← valid when $|x|<1$

$$-\ln(1-x) = \int \frac{dx}{1-x} = \sum_{n=0}^{\infty} \int x^n dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

to figure out C, plug in $x=0$ to see $-\ln(1-0) = C \Rightarrow C=0$.

Thus, $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. Valid when $|x|<1$

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Ex: Find a power series representation for $\ln(1+x)$.

We know a power series representation for $\ln(1-x)$. Replace x by $-x$ to get

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+2} x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

Valid when $|x|<1$.

Ex: Find a power series representation for $\arctan(x)$.

Sol'n: Recall

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Let's work backwards. When $|x|<1$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan(x) = \int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Plug in $x=0$ to see $\arctan(0) = C$

$0 = C$

Therefore $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ when $|x|<1$

Evaluate $\int \frac{dx}{1+x^3}$ using power series. Approximate $\int_0^{0.5} \frac{dx}{1+x^3}$.

Sol'n: Note that

$$\frac{1}{1+x^3} = \cancel{\frac{1}{1-(-x)^3}} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n \quad \text{valid when } |x^3|<1$$
$$|x|<1.$$

So $$\int \frac{dx}{1+x^3} = C + \sum_{n=0}^{\infty} (-1)^n \int x^{3n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \quad \text{valid when } |x|<1.$$

To approximate the integral.

$$\int_0^{0.5} \frac{dx}{1+x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (0.5)^{3n+1}}{3n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{3n+1}}{3n+1}$$
$$= \underbrace{(0.5)}_{n=0} - \underbrace{\frac{(0.5)^4}{4}}_{n=1} + \underbrace{\frac{(0.5)^7}{7}}_{n=2} - \frac{(0.5)^{10}}{10} + \cdots$$
$$\doteq 0.48540194\ldots$$

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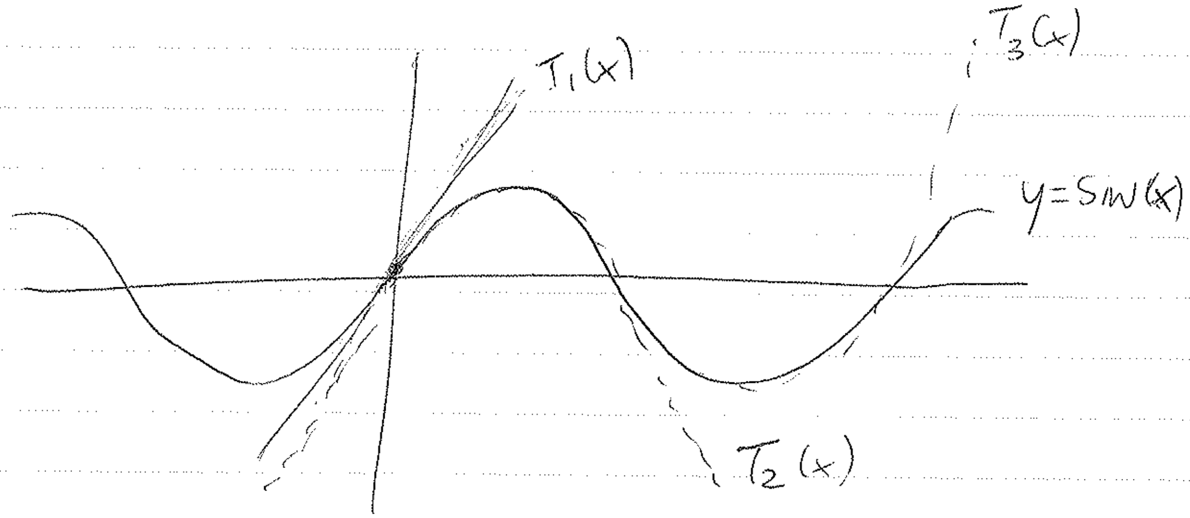
Taylor (& Maclaurin) Series



Idea: Write $f(x)$ as a polynomial (ie a power series).

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \quad (\star)$$

To find c_0, plug in $x=a$. We get $\boxed{f(a) = c_0}$

To find c_1 differentiate $(\star)$ to see that

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Plug in $x=a$ to get $c_1 = f'(a)$.

To find c_2, differentiate again

$$f''(x) = 2c_2 + 3 \cdot 2 \cdot c_3(x-a) + \cdots$$

Plug in $x=a$, $c_2 = f''(a)/2$.

to find c_3, differentiate, plug in $x=a$ to get $c_3 = f'''(a)/3!$

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In general,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (\square)$$

where $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$.

Theorem: If f has a power series representation, it takes the form $(\square)$.

In our course *all functions we care about have power series representations and so the theorem holds for our functions.

If $a=0$, Taylor series are sometimes Maclaurin Series.

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Taylor Series

n^{th} derivative.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series centred at a.

Ex: Find the Taylor Series Expansion about 0 of $f(x) = e^x$ and find the ~~interg~~ interval of convergence.

Sol'n: If $f(x) = e^x$ then $f^{(n)}(x) = e^x$. So, $f^{(n)}(0) = e^0 = 1$. For all $n \geq 0$. Thus, the Taylor series expansion for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

To find interval of convergence, we want to know for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge. Use ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \quad \text{for any } x.$$

Since $0 < 1$, the ratio test says that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real values of x. Hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all real } x.$$

Note. The same proof shows for any $a \in \mathbb{R}$, we have

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n.$$

NOTATION

$0^0 = 1$

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Proposition: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Sol'n: We find the Taylor Series expansion of $\sin(x)$ about 0.

$$\begin{aligned} f(x) &= \sin(x) & f(0) &= 0 \\ f'(x) &= \cos(x) & f'(0) &= 1 \\ f''(x) &= -\sin(x) & f''(0) &= 0 \\ f'''(x) &= -\cos(x) & f'''(0) &= -1 \\ f''''(x) &= \sin(x) & f''''(0) &= 0 \end{aligned}$$

These values are cyclic of order 4. Hence the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

The ratio test again shows this is valid everywhere. ☒

Prop'n: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Sol'n: Since $\frac{d}{dx}\sin(x) = \cos(x)$, we have

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \checkmark$$

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Ex: Find the Taylor Series expansion about 0 of $x^2\cos(x^3)$

Sol'n: $x^2\cos(x^3) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}$

Ex: Find the Taylor Series expansion of $\sin(x)\cos(x)$ about 0 (first 3 non-zero terms)

$$\sin(x)\cos(x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= x - \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^5}{2!\,3!} + \frac{x^5}{4!} \cdots$$

$$= x - \left(\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{5!} + \frac{1}{2!\,3!} + \frac{1}{4!}\right)x^5 \cdots$$

Table of expansions:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}$

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad x \in \mathbb{R}$ $\qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad x \in \mathbb{R}$

$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad |x|<1$ $\qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad |x|<1$

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Applications of Taylor Polynomials

Ex: Find a closed form for the sum

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \dots = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \dots \quad (\ast)$$

Sol'n: Note that

$$(\ast) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\tfrac{1}{2}\right)^n = \ln\left(1+\tfrac{1}{2}\right) = \ln\left(\tfrac{3}{2}\right)$$

Ex: Let $c \in \mathbb{R}$. Compute $\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{c}{n}\right)\right)$.

Sol'n: Use Taylor Series.

$$\begin{aligned}
\lim_{n\to\infty} n^2\left(1-\cos\left(\tfrac{c}{n}\right)\right) &= \lim_{n\to\infty} n^2\left(1-\left(1-\frac{\left(\frac{c}{n}\right)^2}{2!} + \frac{\left(\frac{c}{n}\right)^4}{4!} - \frac{\left(\frac{c}{n}\right)^6}{6!} + \dots\right)\right) \\
&= \lim_{n\to\infty} n^2\left(\frac{c^2}{n^2\cdot 2!} - \frac{c^4}{n^4\cdot 4!} + \frac{c^6}{n^6\cdot 6!} - \dots\right) \\
&= \lim_{n\to\infty}\left(\frac{c^2}{2} - \frac{c^4}{n^2\cdot 24} + \frac{c^6}{n^4\cdot 720} - \dots\right) \\
&= \frac{c^2}{2}.
\end{aligned}$$

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Taylor Polynomials:

Recall: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

We can use Taylor polynomials to approximate ~~functions~~ functions.

$T_0(x) = 0$

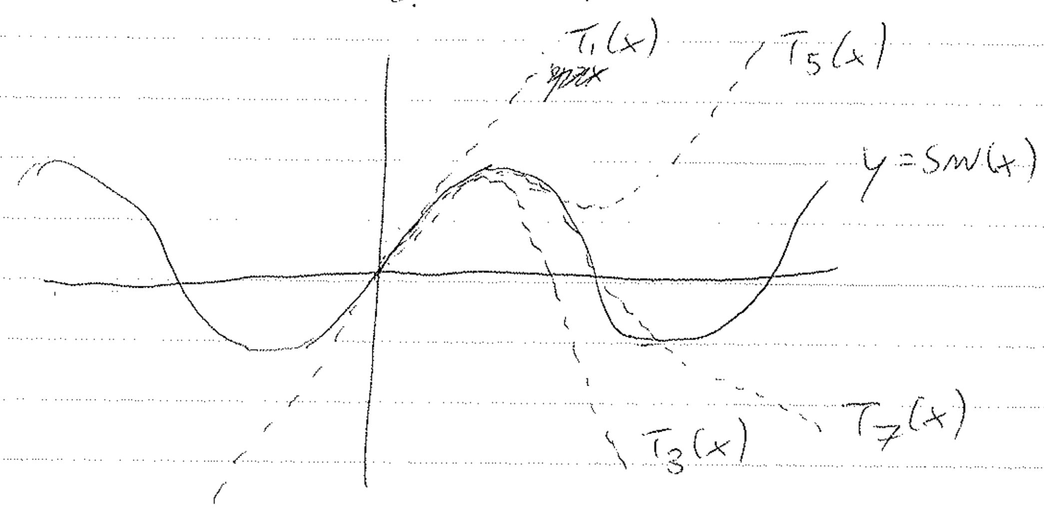
$T_1(x) = x$

$T_2(x) = x$ (no x^2 term)

$T_3(x) = x - \frac{x^3}{3!} = T_4(x)$

$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$T_7(x) = x - x^3/3! + x^5/5! - x^7/7!$



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Taylor Series and Differential Equations.

Let's Solve $\frac{dy}{dx} = y$ with $y(0) = 1$ using Taylor series.

(Recall: we know already that the solution is $y = e^x$)

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Since $y(0) = 1$, we know that $a_0 = 1$. Differentiating gives

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Since $\frac{dy}{dx} = y = a_0 + a_1 x + a_2 x^2 + \cdots$ we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

We compare coefficients. This gives

$$1 = a_0 = a_1 \qquad \text{so} \quad a_1 = 1.$$

Compare coefficients of x gives

$$1 = a_1 = 2a_2 \qquad \text{so} \quad a_2 = \tfrac{1}{2}.$$

Compare coefficients of x^2 gives

$$\tfrac{1}{2} = a_2 = 3a_3 \qquad \text{so} \quad a_3 = \tfrac{1}{6} = \tfrac{1}{3!}$$

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If we keep going, we see that $a_4 = \frac{1}{4!}$ and in general $a_n = \frac{1}{n!}$

This gives

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x.$$ is the solution.

- - - —— END OF COURSE - - - - -