Summations

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Summation Review

Recall some of the notation from class

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=3}^{12} 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 20$$

$$\sum_{k=1}^{n} k = 1 + 2 + ... + n$$

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$

Aside: Change of variable

Notice that

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

and

$$\sum_{j=2}^{5} (j-1) = ((2)-1) + ((3)-1) + ((4)-1) + ((5)-1)$$
$$= 1 + 2 + 3 + 4 = 10$$

are equal. To see this without writing it out, we do a change of variables, in this case replacing i with $j-1$ so that $i = j-1$. When we do this, we also need to change the indices. The lower bound changes from 1 to 2 and the upper bound changes from 4 to 5.

Summation Formulas

Our goal will be to show that

$$\sum_{i=1}^{n} i = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

and further that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Formula 1

To show the first formula, we will use a technique commonly cited as Gauss' technique. As the story goes, Gauss' grade school teacher made him, as a punishment, add up all the numbers from 1 to 100. Gauss cleverly noticed that you could do this very quickly by adding the sum twice and pairing numbers like so

$$\begin{array}{lll} 1 + 2 + \ldots & +100 & \\ +100 + 99 + \ldots & +1 & \\ \hline 101 + 101 + \ldots & +101 & \text{100 times} \end{array}$$

In this manner, Gauss saw that adding the sum twice gave a sum of (101)(100). Since he took the sum twice, the value he wanted was precisely $\frac{(101)(100)}{2} = 5050$.

Formula 1 Proof

We're going to do the same trick except we will use our sigma notation. First, let $S := \sum_{i=1}^{n} i$. Then, notice that

$$2S = S + S = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i = \sum_{i=1}^{n} i + \sum_{i=1}^{n}((n+1) - i)$$
$$= \sum_{i=1}^{n}(i + n + 1 - i) = \sum_{i=1}^{n}(n+1) = n(n+1)$$

and then isolating for S gives us that $S = \frac{n(n+1)}{2}$.

Formula 1 Another Proof

Let's prove this again in a way that generalizes quite nicely. Lets examine the sum of squares starting with 0.

$$\begin{aligned}\sum_{i=0}^{n} i^2 &= \sum_{i=0}^{n}(i+1)^2 - (n+1)^2 \\ &= \sum_{i=0}^{n}(i^2+2i+1) - (n+1)^2 \\ &= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1 - (n^2+2n+1)\end{aligned}$$

Notice that the sum of i^2 appears identically on both sides of the equation so we may cancel those terms to get

$$0 = 2\sum_{i=0}^{n} i + n + 1 - n^2 - 2n - 1 = 2\sum_{i=0}^{n} i - n^2 - n$$

and then isolating for the sum gives us that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Formula 2 Proof

Using a similar idea as above, lets try the same proof but starting with a sum of cubes starting at 0.

$$\begin{aligned}\sum_{i=0}^{n} i^3 &= \sum_{i=0}^{n}(i+1)^3 - (n+1)^3 \\ &= \sum_{i=0}^{n}(i^3+3i^2+3i+1) - (n+1)^3 \\ &= \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1 - (n^3+3n^2+3n+1)\end{aligned}$$

Notice that the sum of i^3 appears identically on both sides of the equation so again we may cancel those terms to get...

Formula 2 Proof

$$\sum_{i=0}^{n} i^3 = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1 - (n^3 + 3n^2 + 3n + 1)$$

... gives

$$\begin{aligned} 0 &= 3\sum_{i=0}^{n} i^2 + \frac{3n(n+1)}{2} + n + 1 - n^3 - 3n^2 - 3n - 1 \\ &= 3\sum_{i=0}^{n} i^2 - \frac{n(n+1)(2n+1)}{2} \end{aligned}$$

and then isolating for the sum gives us that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercises

As an exercise, try to compute the formula

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}.$$

Also, try to evaluate

$$\sum_{i=5}^{15} i \qquad \text{and} \qquad \sum_{i=3}^{10} i^2$$