MATHEMATICS 101 Section 211
Quiz #9, April 2, 2012

Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

1) For each of the following power series, determine the radius of convergence and the interval of convergence. (4 points each)

(i) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n3^n}{n^3+5}$

Solution: Using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{x^{n+1}3^{n+1}}{(n+1)^3+5}(-1)^n\frac{n^3+5}{x^n3^n}\right| \\ &= \lim_{n\to\infty}3|x|\frac{n^3+5}{(n+1)^3+5} \\ &= \lim_{n\to\infty}3|x|\frac{n^3(1+\frac{5}{n^3})}{n^3(1+\frac{1}{n})^3+5} \\ &= \lim_{n\to\infty}3|x|\frac{n^3(1+\frac{5}{n^3})}{n^3((1+\frac{1}{n})^3+\frac{5}{n^3})} \\ &= \lim_{n\to\infty}3|x|\frac{1+\frac{5}{n^3}}{(1+\frac{1}{n})^3+\frac{5}{n^3}} \\ &= 3|x|\end{aligned}$$

The ratio test tells us that the series diverges when $3|x|>1$ and that the series converges when $3|x|<1$ so when $|x|<\frac{1}{3}$. This tells us that $R=\frac{1}{3}$. Thus, we need only to check convergence at the endpoints, so when $x=\pm\frac{1}{3}$. At $x=\frac{-1}{3}$, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{-1}{3})^n3^n}{n^3+5}=\sum_{n=0}^{\infty}\frac{1}{n^3+5}<\sum_{n=0}^{\infty}\frac{1}{n^3}<\infty$$

with the second to last line above holding since $n^3<n^3+5$ and the final statement about convergence holds by the p-test. Hence by the comparison test (valid since all the terms wind up being positive in this case), we have convergence at $x=\frac{-1}{3}$. At $x=\frac{1}{3}$, notice that

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{3})^n3^n}{n^3+5}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^3+5}$$

and this is absolutely convergent by the previous part and so this must also converge. Hence the interval of convergence includes both endpoints and is $I=[-\frac{1}{3},\frac{1}{3}]$. ■

Marking Scheme: Two marks for correctly determining the radius of convergence. The last two marks for getting the interval of convergence correct (basically one for each endpoint). If the answers are not stated coherently (but you feel like they are correct), then only remove one mark for a lack of coherency. The things $R=\frac{1}{3}$ and $I=[-\frac{1}{3},\frac{1}{3}]$ with proof are what I'm looking for. How students justify the series at the endpoints converges can be done in many ways, but I think the above is the easiest way.

(ii) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x+3)^n}{\sqrt{n}+2}$

Solution: Using the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(2x+3)^{n+1}}{\sqrt{n+1}+2}(-1)^n\frac{\sqrt{n}+2}{(2x+3)^n}\right| \\
&= \lim_{n\to\infty}|2x+3|\frac{\sqrt{n}+2}{\sqrt{n+1}+2} \\
&= \lim_{n\to\infty}|2x+3|\frac{\sqrt{n}(1+\frac{2}{\sqrt{n}})}{\sqrt{n}\sqrt{1+\frac{1}{n}}+5} \\
&= \lim_{n\to\infty}|2x+3|\frac{\sqrt{n}(1+\frac{2}{\sqrt{n}})}{\sqrt{n}(\sqrt{1+\frac{1}{n}}+\frac{5}{\sqrt{n}})} \\
&= \lim_{n\to\infty}|2x+3|\frac{1+\frac{2}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+\frac{5}{\sqrt{n}}} \\
&= |2x+3|
\end{aligned}$$

The ratio test tells us that the series diverges when $|2x+3|>1$ and that the series converges when $|2x+3|<1$ so when $|x+\frac{3}{2}|<\frac{1}{2}$. This gives the radius of convergence to by $R=\frac{1}{2}$. To find the interval of convergence, we just need to check the values where $x=-\frac{1}{2}-\frac{3}{2}=-2$ and $x=\frac{1}{2}-\frac{3}{2}=-1$. For $x=-2$, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2(-2)+3)^n}{\sqrt{n}+2}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}+2}$$

Now, notice that $\sqrt{n}+2<2\sqrt{n}$ when $n\geq 4$ and so

$$\frac{1}{2\sqrt{n}}<\frac{1}{\sqrt{n}+2}$$

Now, by the p-series test, we know that

$$\sum_{n=0}^{\infty}\frac{1}{2\sqrt{n}}$$

diverges and so by the comparison test, we get that

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}+2}$$

also diverges. Lastly, at $x=-1$, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2(-1)+3)^n}{\sqrt{n}+2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}+2}$$

The terms of this series (in absolute value) are decreasing and limit to 0. Hence by the alternating series test, this sum converges. So we include the endpoint $x=-1$ but not $x=-2$ and see that the interval of convergence is $I=(-2,-1]$ completing the proof. ■

Marking scheme: The same as the previous question.

2) Suppose we know that the power series $\sum_{n=0}^{\infty} c_n x^n$ converges at the point $x=-6$. What can we say (if anything) about the convergence of the series below? Explain. (1 mark each)

(i) $\sum_{n=0}^{\infty} c_n 3^n$

Solution: This series must converge. We know from the fact that

$$\sum_{n=0}^{\infty} c_n x^n$$

converges at $x = 6$ and our power series is centred at 0, then we have that our radius of convergence is $R = 6$ and our interval of convergence contains the interval $J := [-6, 6)$ (it could be larger but no smaller). This followed from a theorem about power series in class. Hence, as $3 \in J$, we know that our series above must converge. ■

(ii) $\sum_{n=0}^{\infty} c_n 6^n$

Solution: As the reasoning above showed, the interval of convergence must contain the interval $J := [-6, 6)$. However, we actually get no information about the value at the point $x = 6$. It could converge or diverge. The following two examples show both behaviours. The first series converges on $[-6, 6]$.

$$\sum_{n=0}^{\infty} \frac{x^n}{6^n n^2}$$

and this next series converges on $[-6, 6)$

$$\sum_{n=0}^{\infty} \frac{x^n}{6^n \sqrt{n}}.$$

Both of these facts I strongly urge you verifying to see what I mean. In any case, we cannot conclude anything about the series in this part of the question. ■

Marking Scheme: The answers pretty much have to be similar to this. Giving examples for the first part without explaining a formal proof that discusses the radius and interval of convergence should garner 0/1. For the second problem - they can either explain it by finding the examples as I have done or by giving the reason I gave before the examples. The key is knowing that power series have a radius of convergence and that the question tells us something about the size of this radius.