MATHEMATICS 101 Section 211
Quiz #8, March 26, 2012

Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

1) For each of the following sums, determine if the series is absolutely convergent, conditionally convergent, or divergent. Correct answers not properly justified will earn no points. (3 points each)

(i) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$

Solution: Taking the absolute value of the terms and examining this series means we are looking at the series

$$\sum_{n=2}^{\infty}\frac{1}{\ln n}$$

For $n \geq 1$ notice that $n > \ln(n)$.

NB: the following argument can be omitted and is included only for complete rigour

This holds since if we look at $f(x) = \ln(x) - x$, then $f'(x) = \frac{1}{x} - 1$ and so this derivative is negative when

$$f'(x) = \tfrac{1}{x} - 1 < 0$$

that is, when $\frac{1}{x} < 1$ and so $1 < x$. Since at $x = 1$, we have that $f(1) = \ln(1) - 1 = -1 < 0$ and since the series is decreasing when $x > 1$, we have that $f(x) < 0$ when $x \geq 1$ and so

$$\ln(x) - x = f(x) < 0$$

that is $\ln(x) < x$ when $x \geq 1$.

The rest must be included. Hence,

$$\sum_{n=2}^{\infty}\frac{1}{n} < \sum_{n=2}^{\infty}\frac{1}{\ln n}$$

and since the sum on the left diverges (it is the harmonic series, so it diverges by the p-series test) the sum on the right diverges by the comparison test. Thus the series is not absolutely convergent. Notice that

$$\lim_{n\to\infty}\frac{1}{\ln n} = 0.$$

The terms satisfy $\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$ since of course $\ln n < \ln(n+1)$. Hence, the terms are decreasing. Thus, the alternating test says that

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$$

converges. Hence our series conditionally converges.

Marking Scheme: It is enough to note that the harmonic series is the harmonic series and hence is divergent (student's need not write that this diverges by the p-series test). One mark for noting that this series converges by the alternating series test. Two marks for justifying that the series is NOT absolutely convergent (so one for noticing the harmonic series trick and one for justifying this trick via the comparison test). Students MUST justify that their series is not absolutely convergent in order to receive most of the marks. (This is where the work really is)

(ii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2 4^n}{n!}$

Solution: Applying the ratio test, we see that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2 4^{n+1}}{(n+1)!} \div \frac{n^2 4^n}{n!}\right| \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \cdot \frac{n!}{(n+1)!} \cdot \frac{4^{n+1}}{4^n} \\ &= \lim_{n\to\infty}(1+\tfrac{1}{n})^2 \cdot \frac{1}{n+1} \cdot 4 \\ &= 0\end{aligned}$$

Hence, the ratio test tells us that this series converges absolutely since $0 < 1$.

Marking Scheme: Again I can't see any other solution so award one mark for realizing this is a ratio test question. One mark for the limit and one mark for the correct final justification via the statement of the ratio test.

(iii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{7n}{10n+7}$

Solution: Taking the limit of the terms yields

$$\lim_{n\to\infty}(-1)^n\frac{7n}{10n+7} = \lim_{n\to\infty}(-1)^n\frac{7}{10+\frac{7}{n}}$$

and this sequence diverges since there is a subsequence tending to $\frac{7}{10}$ and one tending to $\frac{-7}{10}$ (namely the even n terms and the odd n terms respectively). Hence, the divergence test tells us that this series diverges.

Marking Scheme: You may award part marks for using the alternating series test (so long as they realize that it failed). A full mark solution uses the divergence test as above. No other test should solve this problem (though I could be mistaken...). Students must state which test they are using and should give at least some justification as to why the limit of the sequence of terms diverges.

2) In the next lecture, we will see that a power series expansion of e^x is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Explain how you would use this expansion to find an n such that if you add up the first n terms in the power series expansion of e^{-2}, you are correct up to three decimal places. You do not need to compute this value of n. Only to explain what theorems and methods you are using to find this n. (1 point)

Solution: To do this, we use the alternating series test estimation. Since

$$e^x = \sum_{n=0}^{\infty}\frac{(-2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!}$$

is an alternating series that satisfies the alternating series test (terms limit to 0 and the terms are decreasing), we can use the alternating series estimation theorem and try to find an n so that

$$\frac{2^n}{n!} < 0.0001$$

where here we use an extra decimal place above to account for cases like if our numerical answer was 0.3998. We then check our answer to make sure that up to this error, we are still correct to three decimal places.

Marking Scheme: I'm not sure what to expect here. I really just want students to get used to explaining their ideas. So one mark for anything that is reasonable and along the lines of using the alternating series estimation theorem. Be generous and give marks according to if the argument is well laid out and coherent.