MATHEMATICS 101 Section 211
Quiz #7, March 12, 2012
Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **UBC Stud. No.:**

1) State the Monotone Convergence Theorem (also known as the Monotone Sequence Theorem in the textbook). (2 points)

Solution: Every bounded monotonic sequence converges. ■

Marking Scheme: Two points for the correct statement. One for bounded and one for monotonic. If they write "increasing or decreasing" sequence then this is the same as monotonic so award this point. If they only write increasing or they write decreasing then only half a mark for this point.

2) True or false. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} a_n = 0$. Is it true that $\sum_{n=1}^{\infty} a_n$ converges? Give a proof if you think this is true or give a counterexample and a brief description as to why your sum in your counterexample diverges. (2 points)

Solution: This is false. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a counter example. It diverges for example by the p series test, the integral test, or by noting that $s_{2^n} > 1 + \frac{n}{2}$ for all integers n where s_N is the Nth partial sum of the harmonic series. ■

Marking Scheme: Writing false is 0.5 marks. Writing false with a correct counterexample is worth somewhere between 0.5 and 1 additional mark depending on the answer and description given. Giving an intuitive reason why the sum diverges (something like the harmonic series grows like $\ln(n)$ and hence diverges) is half a mark. Writing false, giving a counter example, and a valid brief correct reason is worth a full 2 marks. There are many answers to this question - most should look like the harmonic series above.

3) Evaluate $\sum_{n=2}^{\infty} \frac{3^n}{5^{n+1}}$. (2 points)

Solution: Notice that

$$\sum_{n=2}^{\infty} \frac{3^n}{5^{n+1}} = \frac{1}{5}\sum_{n=2}^{\infty}\left(\frac{3}{5}\right)^n = \frac{1}{5}\cdot\frac{\frac{9}{25}}{1-\frac{3}{5}} = \frac{9}{50}$$

where we used the formula for the sum of a geometric series. The first term occurs when $n = 2$ and so $a = \frac{3^2}{5^2} = \frac{9}{25}$ and the ratio is simply $\frac{3}{5}$. This completes the question. ■

Marking Scheme: Half a mark for pulling out a $\frac{1}{5}$, one mark for a correct application of the geometric series formula (so getting the a and r values correct). Half a mark for the answer.

4) Evaluate $\lim_{n\to\infty} \frac{e^n}{n^n}$. (2 points)

Solution: Intuitively, this should converge since the numerator grows much slower than the denominator. To show this we will use the squeeze theorem. Notice that each of the above terms are positive (so bounded below by 0). Further

$$\frac{e^n}{n^n} = \frac{e}{n}\cdot\frac{e}{n}\cdot\frac{e}{n}\cdot\frac{e}{n}\cdot\ldots\cdot\frac{e}{n}$$

Now, notice that for all $n \geq 3$, we have $e < n$ and that $\frac{e}{n} < \frac{e}{3}$ for all $n \geq 4$. Hence

$$\begin{aligned} \frac{e^n}{n^n} &= \frac{e}{n} \cdot \frac{e}{n} \cdot \frac{e}{n} \cdot \frac{e}{n} \cdot \ldots \cdot \frac{e}{n} \\ &< \left(\frac{e}{3}\right)^n \end{aligned}$$

Notice that

$$\lim_{n\to\infty} \left(\frac{e}{3}\right)^n = 0$$

and so by the squeeze theorem, we have that

$$\lim_{n\to\infty} \frac{e^n}{n^n} = 0$$

as claimed.

Solution 2: (Modified from Justin Li) - Notice that for $n \geq 3$, we have that $3^n \geq n^n$. Hence

$$0 \leq \frac{e^n}{n^n} \leq \frac{e^n}{3^n} \leq \left(\frac{e}{3}\right)^n$$

Notice that

$$\lim_{n\to\infty} \left(\frac{e}{3}\right)^n = 0$$

and so by the squeeze theorem, we have that

$$\lim_{n\to\infty} \frac{e^n}{n^n} = 0$$

as claimed.

■

Marking Scheme: The only solutions I can think of off the top of my head are the ones above. Award one point for some reasonable attempt, trying to use the squeeze theorem in some way etc. Two marks for a correct solution. Minus 0.5 if they don't bound the sequence below by 0 (ie apply the squeeze theorem incorrectly).

5) Give examples of two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ consisting of all positive terms so that $\lim\limits_{n\to\infty} (-1)^n a_n$ converges and $\lim\limits_{n\to\infty} (-1)^n b_n$ diverges.

Solution: *NOTE:* I actually made a mistake in the wording of this problem. I wanted the sequence of b_n values to also converge. No big deal but just know ti was meant to be slightly harder. Let $a_n = \frac{1}{n}$ and $b_n = 1$. Then

$$\lim_{n\to\infty} (-1)^n a_n = \lim_{n\to\infty} \frac{(-1)^n}{n}$$

Since

$$\lim_{n\to\infty} |(-1)^n a_n| = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

we know that

$$\lim_{n\to\infty} (-1)^n a_n = \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

and hence converges. For the second example, notice that

$$\lim_{n\to\infty} (-1)^n b_n = \lim_{n\to\infty} (-1)^n.$$

So take the subsequence $c_n = b_{2n-1} = -1$ for all $n \geq 1$ and the subsequence $d_n = b_{2n} = 1$ for all $n \geq 1$. Notice that these two subsequences are infinite and they tend to different values. Hence, the sequence cannot converge. This completes the question. ■

Marking Scheme: Notice that the a_n terms HAVE to be a sequence tending to 0. The second example can be a lot of different things. I am not asking a proof so they do not need to supply one. One mark for each sequence if it is correct.