MATHEMATICS 101 Section 211
Quiz #6, March 5, 2012

Show all your work. Use back of page if necessary. Calculators are not allowed.
If you are asked to evaluate a definite integral and the integral is divergent, show that it is divergent.

Last Name: **First Name:** **UBC Stud. No.:**

1) Find the centroid (centre of mass) of the curve $f(x) = e^x$ between the x-axis on the interval $0 \leq x \leq 1$. (4 points)

Solution: We compute the values. Notice that

$$A = \int_0^1 e^x\,dx = e^1 - e^0 = e - 1.$$

The x-coordinate of the centre of mass is

$$\bar{x} = \frac{1}{e-1}\int_0^1 xe^x\,dx$$

To solve this, we use integration by parts. Let $u = x$ and $dv = e^x dx$ so that

$$\begin{aligned} u &= x & v &= e^x \\ du &= dx & dv &= e^x dx \end{aligned}$$

Applying gives

$$\begin{aligned}
\bar{x} &= \frac{1}{e-1}\int_0^1 xe^x\,dx \\
&= \frac{1}{e-1}\left(xe^x\Big|_0^1 - \int_0^1 e^x\,dx \right) \\
&= \frac{1}{e-1}\left((1)e^1 - (0)e^0 - \left(e^x\Big| + 0^1 \right) \right) \\
&= \frac{1}{e-1}\left(e - (e^1 - e^0) \right) \\
&= \frac{1}{e-1}(e - e + 1) = \frac{1}{e-1}
\end{aligned}$$

The y coordinate is

$$\begin{aligned}
\bar{y} &= \frac{1}{2e-2}\int_0^1 e^{2x}\,dx \\
&= \frac{1}{2e-2}\left(\frac{e^{2x}}{2}\Big|_0^1 \right) \\
&= \frac{1}{2e-2}\left(\frac{e^{2(1)}}{2} - \frac{e^{2(0)}}{2} \right) \\
&= \frac{1}{2e-2}\left(\frac{e^2}{2} - \frac{1}{2} \right) \\
&= \frac{e^2-1}{4e-4}
\end{aligned}$$

and hence the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e^2-1}{4e-4} \right)$$

Marking Scheme: One mark for the correct formulas and final answer. One mark for A, one mark for $\bar{x}$, one mark for $\bar{y}$

2) Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{x}$. (3 points)

Solution: Notice that our function is not defined at $x = 0$. Hence we need to break up this improper integral. Notice that

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x} &= \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} \\
&= \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} \\
&= \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{x} + \lim_{a \to 0^+} \int_{a}^{1} \frac{dx}{x} \\
&= \lim_{b \to 0^-} \ln|x|\Big|_{-1}^{b} + \lim_{a \to 0^+} \ln|x|\Big|_{a}^{1} \\
&= \lim_{b \to 0^-} (\ln|b| - \ln|-1|) + \lim_{a \to 0^+} (\ln|1| - \ln|a|) \\
&= \lim_{b \to 0^-} (\ln|b| - \ln|-1|) + \lim_{a \to 0^+} (\ln|1| - \ln|a|)
\end{aligned}$$

Now, notice that

$$\lim_{b \to 0^-} \ln|b|$$

does not exist (it approaches infinity). Hence the integral diverges. ■

Marking Scheme: One mark for identifying this is an improper integral, one mark for the integral, one mark for the integral diverging

3) Solve y given that $\dfrac{dy}{dx} = \dfrac{e^{\arctan(x)} \csc(y)}{1 + x^2}$. (3 points)

Solution: This is a separable differential equation. Remembering that $\csc(y) = \frac{1}{\sin(y)}$, we have

$$\begin{aligned}
\sin(y)\, dy &= \frac{e^{\arctan(x)}}{1 + x^2}\, dx \\
\int \sin(y)\, dy &= \int \frac{e^{\arctan(x)}}{1 + x^2}\, dx
\end{aligned}$$

Now, let $u = \arctan(x)$ so that $du = \frac{dx}{1+x^2}$ then

$$\begin{aligned}
\int \sin(y)\, dy &= \int \frac{e^{\arctan(x)}}{1 + x^2}\, dx \\
-\cos(y) &= \int e^u\, du \\
-\cos(y) &= e^u + C \\
-\cos(y) &= e^{\arctan(x)} + C \\
\cos(y) &= -e^{\arctan(x)} - C \\
y &= \arccos(-e^{\arctan(x)} - C)
\end{aligned}$$

completing the question. ■

Marking Scheme: One mark for the separation, one mark for integrating via a substitution (ie getting the integral correct), one mark for getting the final answer.