MATHEMATICS 101 Section 211  
Quiz #6, March 5, 2012

Show all your work. Use back of page if necessary. Calculators are not allowed.

If you are asked to evaluate a definite integral and the integral is divergent, show that it is divergent.

Last Name: First Name: UBC Stud. No.:

1) Find the centroid (centre of mass) of the curve \( f(x) = e^x \) between the \( x \)-axis on the interval \( 0 \leq x \leq 1 \). (4 points)

Solution: We compute the values. Notice that

\[
A = \int_0^1 e^x \, dx = e^1 - e^0 = e - 1.
\]

The \( x \)-coordinate of the centre of mass is

\[
\bar{x} = \frac{1}{e - 1} \int_0^1 xe^x \, dx
\]

To solve this, we use integration by parts. Let \( u = x \) and \( dv = e^x \, dx \) so that

\[
\begin{align*}
    u &= x \\
    v &= e^x \\
    du &= dx \\
    dv &= e^x \, dx
\end{align*}
\]

Applying gives

\[
\bar{x} = \frac{1}{e - 1} \int_0^1 xe^x \, dx
= \frac{1}{e - 1} \left( xe^x \bigg|_0^1 - \int_0^1 e^x \, dx \right)
= \frac{1}{e - 1} \left( (1)e^1 - (0)e^0 - \left( e^x \bigg|_0^1 \right) \right)
= \frac{1}{e - 1} \left( e - (e^1 - e^0) \right)
= \frac{1}{e - 1} \left( e - e + 1 \right) = \frac{1}{e - 1}
\]

The \( y \) coordinate is

\[
\bar{y} = \frac{1}{2e - 2} \int_0^1 e^{2x} \, dx
= \frac{1}{2e - 2} \left( e^{2x} \bigg|_0^1 \right)
= \frac{1}{2e - 2} \left( e^{2(1)} - e^{2(0)} \right)
= \frac{1}{2e - 2} \left( e^2 - 1 \right)
= \frac{e^2 - 1}{4e - 4}
\]

and hence the centroid is

\[
(\bar{x}, \bar{y}) = \left( \frac{1}{e - 1}, \frac{e^2 - 1}{4e - 4} \right)
\]

Marking Scheme: One mark for the correct formulas and final answer. One mark for \( A \), one mark for \( \bar{x} \), one mark for \( \bar{y} \)
2) Evaluate \( \int_{-1}^{1} \frac{dx}{x} \). (3 points)

**Solution:** Notice that our function is not defined at \( x = 0 \). Hence we need to break up this improper integral. Notice that

\[
\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} = \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{x} + \lim_{a \to 0^+} \int_{a}^{1} \frac{dx}{x} = \lim_{b \to 0^-} \ln |x| \bigg|_{-1}^{b} + \lim_{a \to 0^+} \ln |x| \bigg|_{a}^{1} = \lim_{b \to 0^-} (\ln |b| - \ln |1|) + \lim_{a \to 0^+} (\ln |1| - \ln |a|)
\]

Now, notice that

\[
\lim_{b \to 0^-} \ln |b|
\]

does not exist (it approaches infinity). Hence the integral diverges. ■

**Marking Scheme:** One mark for identifying this is an improper integral, one mark for the integral, one mark for the integral diverging.

3) Solve \( y \) given that \( \frac{dy}{dx} = \frac{e^{\arctan(x)} \csc(y)}{1 + x^2} \). (3 points)

**Solution:** This is a separable differential equation. Remembering that \( \csc(y) = \frac{1}{\sin(y)} \), we have

\[
\sin(y) \, dy = \frac{e^{\arctan(x)}}{1 + x^2} \, dx
\]

\[
\int \sin(y) \, dy = \int \frac{e^{\arctan(x)}}{1 + x^2} \, dx
\]

Now, let \( u = \arctan(x) \) so that \( du = \frac{dx}{1 + x^2} \) then

\[
\int \sin(y) \, dy = \int \frac{e^{u}}{1} \, du = e^{u} - \cos(y) = e^{u} + C = e^{\arctan(x)} + C
\]

\[
\cos(y) = -e^{\arctan(x)} - C
\]

\[
y = \arccos(-e^{\arctan(x)} - C)
\]

completing the question. ■

**Marking Scheme:** One mark for the separation, one mark for integrating via a substitution (ie getting the integral correct), one mark for getting the final answer.